

VIBRATIONS OF A TWO SPAN SYMMETRICAL BEAM  
WITH AN ELASTIC MIDSUPPORT UNDER THE ACTION  
OF A CONSTANT, UNIFORMLY MOVING FORCE

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## INTRODUCTION

The study of moving loads and masses over beams has been under investigation for over a hundred years and even at the present time is still the subject of much research.

One finds, for instance, Stokes as early as 1849 solving the case of a single span beam under the action of a constant speed moving mass wherein he neglects the inertia of the beam (10). Kryloff then retained the mass of the beam and neglected the inertia of the moving element, merely considering it as a force (7). This analysis simulates more closely the actual case for moderate speeds. In 1922, Timoshenko worked out the case of a pulsating force moving with constant velocity over a single span simple beam (11). Lowan later worked on cases where the velocity of the load was not constant (8). The most exhaustive works on single span structures can be attributed to Inglis who contributed several papers and in addition wrote a book on the subject (5). Approximate procedures for constant and pulsating forces with mass are developed for single span structures. Also the case of a mass, separated from the beam by a spring, and moving with uniform velocity has been studied by Inglis. Jeffcott also worked on the problem of a simply supported beam under the action of a moving mass (6). His procedure was to first neglect the mass of the moving element, solve the resulting differential equation, and then with this approximate result resubstitute the effects of inertia into the differential equation. In this manner, an iteration process is established which he felt was convergent. However, Steuding has shown that for this problem the process suggested by Jeffcott is only asymptotically convergent (9).



In recent times the problem of a beam with a continuously moving mass has received a great deal of attention as a result of interesting problems arising from the Trans-Arabian pipe lines. Steuding, Ashley, and Housner have published papers on this problem (9, 1, 4).

It will be noted that all this work has been restricted to single span structures. Up to very recent times practically nothing has been published for continuous beams. In 1950, Ayre, Ford, and Jacobsen described the effects of constant and pulsating forces moving with uniform velocity over a symmetrical two span beam with three simple rigid supports (3).

### STATEMENT OF THE PROBLEM

The case of a continuous symmetrical two span beam in which the midsupport is elastic rather than rigid is the subject of this investigation. The disturbance will be a uniformly moving constant force. This will simulate a bridge over a deep ravine wherein the end supports are rigidly supported, although free to move longitudinally, while the midsupport will be somewhat elastic in nature.

In addition, the problem of a symmetrical two span beam with three simple rigid supports, and which is subject to the action of a continuous mass constant in magnitude and moving with uniform speed over the beam is undertaken in appendix 1. This case simulates a pipe line in which fluid is flowing or possibly a two span continuous bridge on which is moving a long constant speed train.

NOTATION

- $E$  . . . Modulus of elasticity  
 $I$  . . . Moment of inertia of the cross-sectional area of the beam about the centroidal axis  
 $\gamma$  . . . Weight density of the beam material  
 $A$  . . . Cross-sectional area of the beam  
 $g$  . . . Acceleration of gravity  
 $a^2$  . . .  $EIg/A\gamma$   
 $p^2$  . . . Separation constant  
 $k^4$  . . .  $p^2/a^2$   
 $l$  . . . Length of one span of the beam  
 $K$  . . . Spring constant of the midsupport of the beam  
 $\mathcal{C}$  . . . Dimensionless parameter  $(4EI/Kl^3)$   
 $X_i$  . . .  $i$ 'th even mode shape  
 $\bar{X}_i$  . . .  $i$ 'th odd mode shape  
 $q_i$  . . .  $i$ 'th normal coordinate for the even modes  
 $\bar{q}_i$  . . .  $i$ 'th normal coordinate for the odd modes  
 $\eta_i$  . . .  $(k_i l)$   
 $\omega_i$  . . .  $(v\eta_i/l)$   
 $\bar{\omega}_i$  . . .  $(i\pi v/l)$   
 $G_i$  . . .  $(\eta_i + \sin\eta_i \cos\eta_i - \cos^2\eta_i \tanh\eta_i - \frac{\cos^2\eta_i}{\cosh\eta_i} (\eta)_i)$   
 $H_i$  . . .  $(\sin\eta_i - 2 \sin\eta_i \cos\eta_i \tanh\eta_i + \cos^2\eta_i \tanh^2\eta_i)$   
 $F_i$  . . .  $(\eta_i - 3 \sin\eta_i \cos\eta_i - 3 \cos\eta_i \tanh\eta_i - \frac{\cos\eta_i}{\cosh\eta_i} (\eta)_i)$

$\alpha_i$  . . . Natural frequency of the i'th even mode

$\bar{\alpha}_i$  . . . Natural frequency of the i'th odd mode

$$\theta_i/\dot{\alpha}_i \cdot \frac{1}{EI} \left[ \frac{1}{(\eta^2 G_i + (4/3)H_i)} \right]$$

$Q_i$  . . . i'th generalized force for the even modes

$\bar{Q}_i$  . . . i'th generalized force for the odd modes

$c$  . . . Distance from the centroidal axis to the outermost fiber of the beam

MATHEMATICAL ANALYSIS OF THE VIBRATIONS OF A TWO SPAN BEAM  
UNDER THE ACTION OF A MOVING LOAD WHEN THE MIDSUPPORT IS ELASTIC

1. Mode Shapes and The Frequency Equations

In this analysis the following assumptions are made -

- a. Deflections are small
- b. Damping is negligible
- c. Rotary inertia and shear effects are negligible
- d. The mass of the load and the midsupport are not considered
- e. The beam is long in comparison to its depth and is simply supported at the ends.

The differential equation for free transverse vibrations of a beam under these restrictions is -

$$EI \frac{\partial^4 y}{\partial x^4} + \frac{A \gamma}{g} \frac{\partial^2 y}{\partial t^2} = 0 \quad (1)$$

Letting  $a^2 = \frac{EIg}{A \gamma}$ , we have -

$$a^2 \frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} = 0 \quad (2)$$

We form a product solution  $y = (X)(T)$  such that  $X$  and  $T$  are functions of displacement and time respectively. Substituting into the differential equation we arrive at -

$$\frac{T''}{T} + a^2 \frac{X'''}{X} = 0$$

Therefore, on separating variables, two ordinary differential equations

with an arbitrary separation constant  $p^2$  are developed.

$$\frac{T''}{T} = -p^2$$

$$\frac{X'''}{X} - \frac{p^2}{a^2} = 0$$

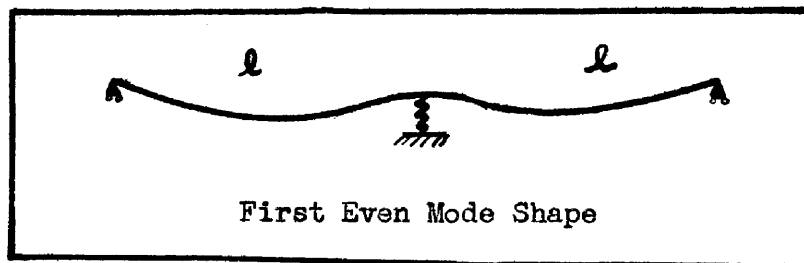
Solving the preceding ordinary differential equations, we find -

$$T = A \sin pt + B \cos pt \quad (3)$$

$$X = C_1 \sin kx + C_2 \cos kx + C_3 \sinh kx + C_4 \cosh kx \quad (4)$$

where  $k^2 = \frac{p^2}{a^2}$

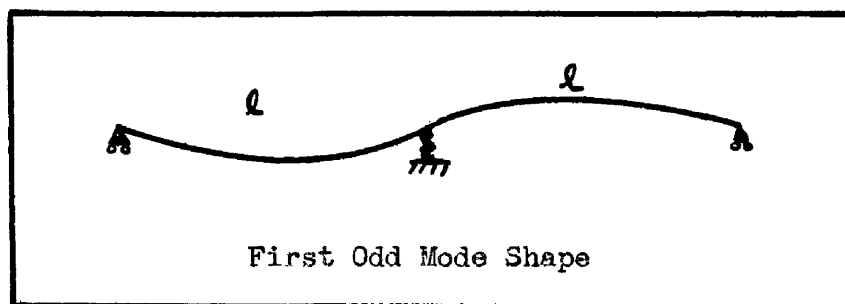
We must now determine the proper solution to fit the boundary conditions of the problem. One sees intuitively that by virtue of the symmetry two basic types of modes exist<sup>1</sup>. The first type, which we will refer to as the even modes, has a zero slope at the midsupport. For these modes there is a deflection possible at the mid-support and the mode shape is affected by the elasticity of this support.




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<sup>1</sup>A mathematical proof of this and succeeding statements will be found in the appendix 3.

The odd modes are simply sinusoids and are unaffected by any elasticity at the midsupport.



The boundary conditions for the even modes are -

A.	$x = 0$	B.	$x = l$	C.	$x = 0$	D.	$x = l$
	$y = 0$		$\frac{dy}{dx} = 0$		$\frac{dy}{dx} = 0$		$Ky = 2EI \frac{d^3y}{dx^3}$

The justification of D can best be seen from strength of materials. The shear diagram for a symmetrical load is shown below with the mid-reaction and shear value at this point indicated.



From A and C one finds  $C_2 = C_4 = 0$  and from B,

$$C_3 = -C_1 \frac{\cos kl}{\cosh kl}$$

On substitution into D we get the transcendental equation to be satisfied -

$$\tan kl - \tanh kl = \frac{4EI}{K} (k)^3 \quad (5)$$

Letting  $kl = \eta$  and  $\frac{4EI}{Kl^3} = \beta$  we finally arrive at the dimensionless equation for the determination of the eigenvalues  $\eta$  -

$$\tanh \eta - \tan \eta = \beta \eta^3 \quad (6)$$

In  $\beta$  among other parameters of the problem is the elasticity of the midsupport. This can range from  $K = 0$  to  $K = \infty$ . In the appendix 2 will be found plots of  $\eta$  vs.  $\beta$  for the first three even modes. Also, there are tables of same following these graphs for more exact computations.

The mode shape or eigenfunction for the  $i$ 'th even mode is given by the equation -

$$X_i = \left[ \sin \eta_i \frac{x}{l} - \frac{\cos \eta_i \sinh \eta_i \frac{x}{l}}{\cosh \eta_i} \right] \quad (7)$$

for the first bay and for the second bay by the equation -

$$X_i = \left[ \sin \eta_i \frac{(l-x)}{l} - \frac{\cos \eta_i \sinh \eta_i \frac{(l-x)}{l}}{\cosh \eta_i} \right] \quad (8)$$

Plots of the mode shapes for the first three even modes are found on the following pages for varying amounts of elasticity of the midsupport. The full lines in all cases are proportional to the deflection while the dotted lines are proportional to the stresses. A detailed analysis for the position of the maximum stress as a function of  $\eta$  for the first two even modes will be found later in the section on resonance.

Note that if  $K = \infty$  then  $\beta = 0$  and the frequency equation becomes  $\tanh \eta - \tan \eta = 0$ . This is the case of a single span, fixed at



# SHAPES OF DEFLECTION AND STRESS CURVES FIRST EVEN MODE

DEFLECTION —  
STRESS - - - -

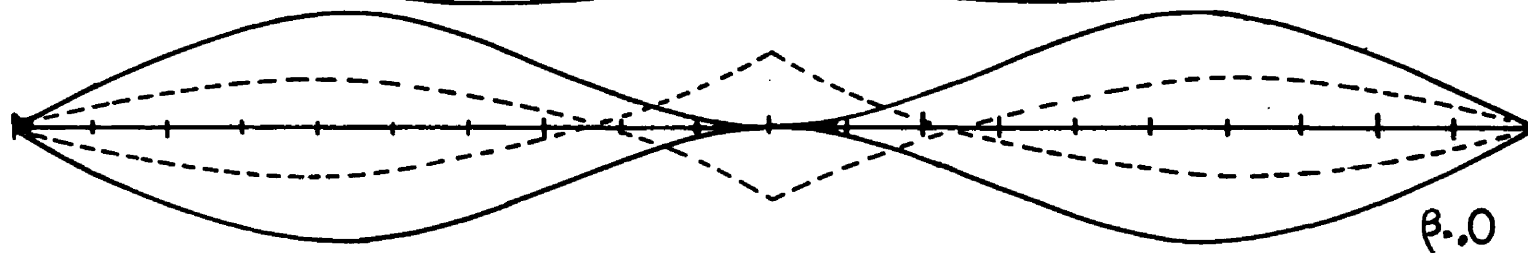
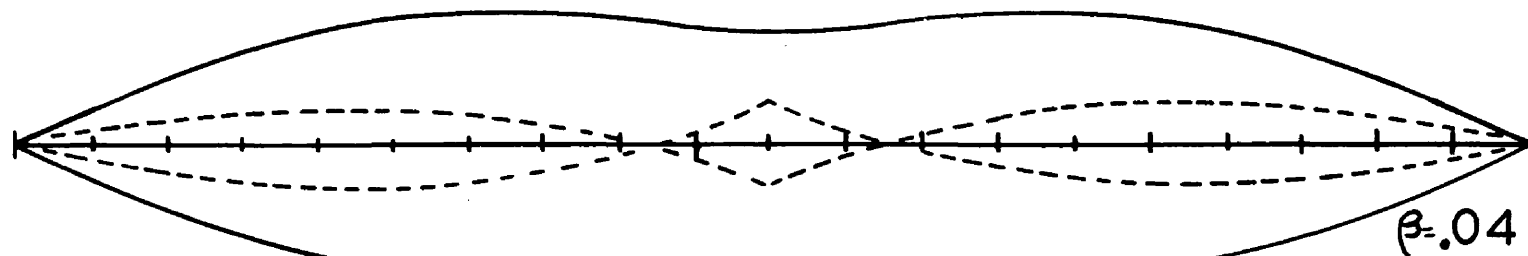
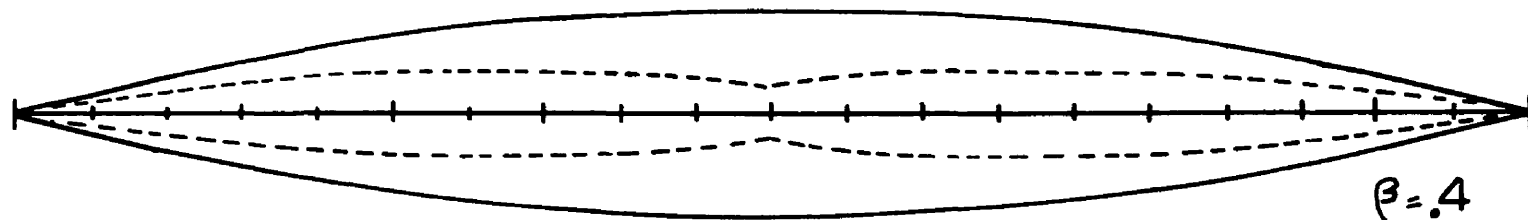
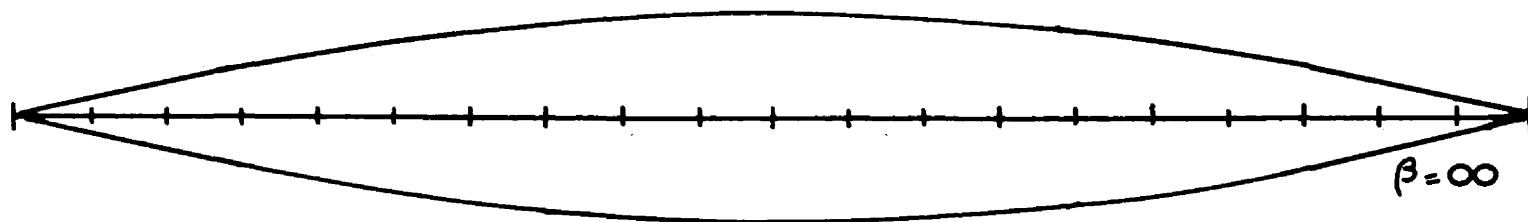
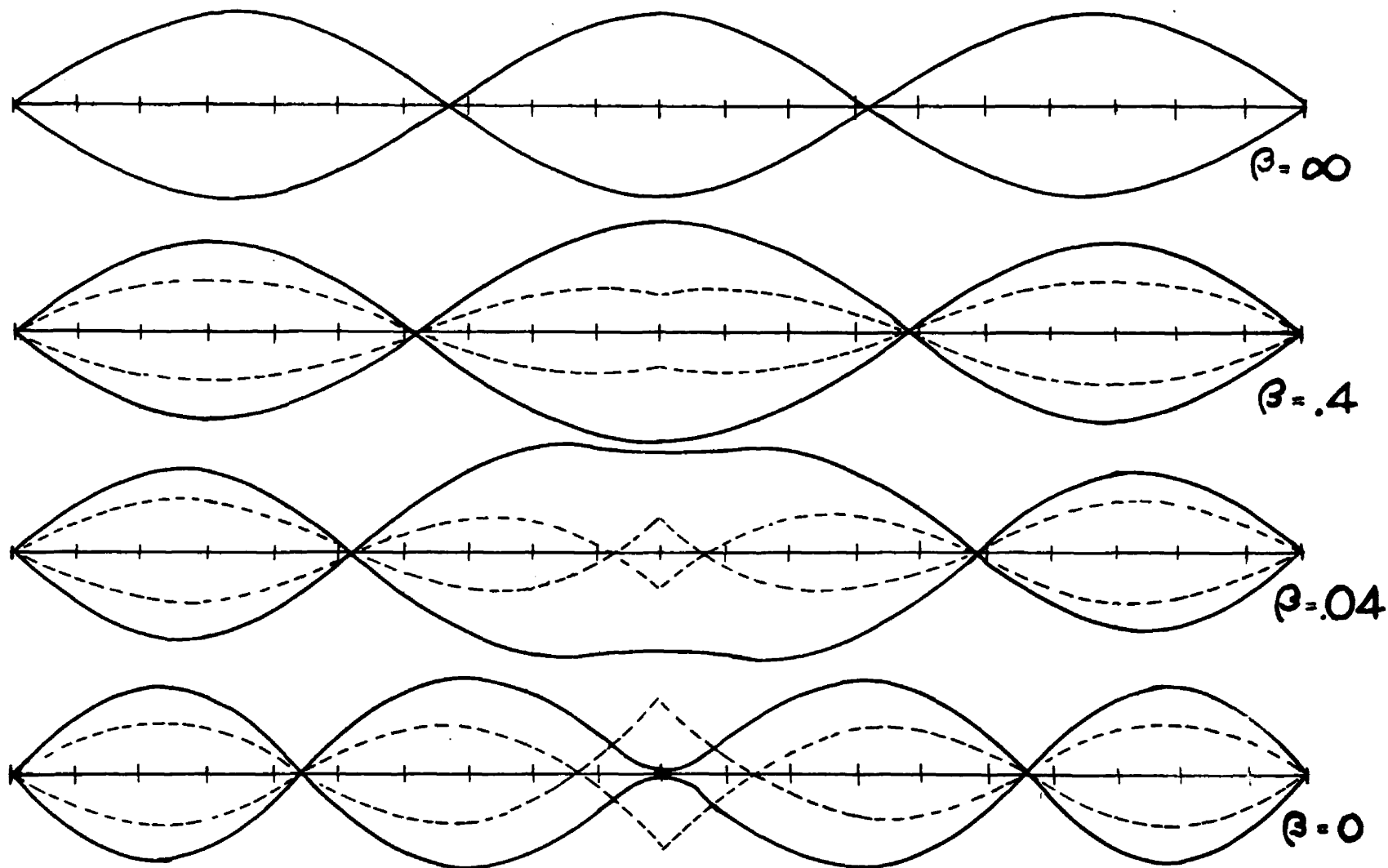


PLATE 1

# SHAPES OF DEFLECTION AND STRESS CURVES SECOND EVEN MODE

DEFLECTION —  
STRESS ----



# SHAPES OF DEFLECTION AND STRESS CURVES THIRD EVEN MODE

DEFLECTION—  
STRESS----

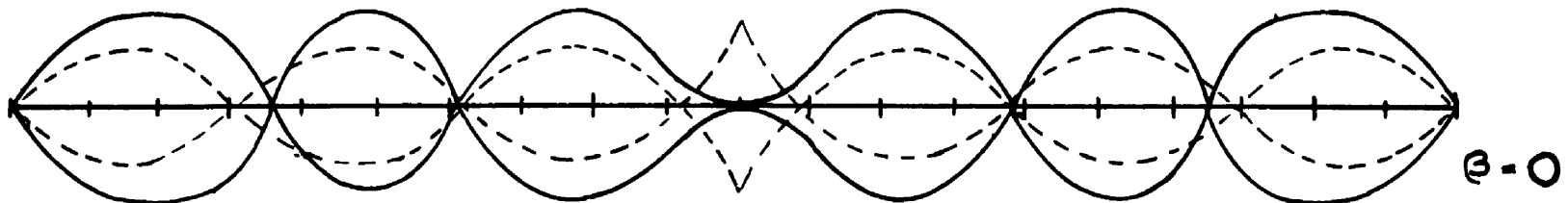
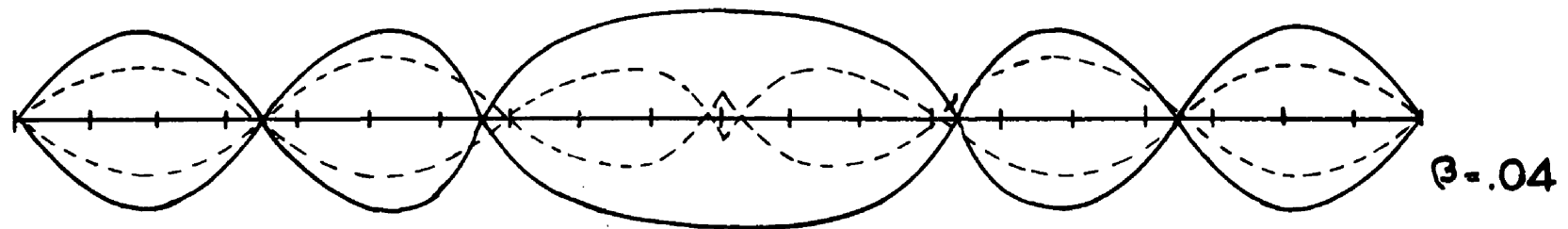
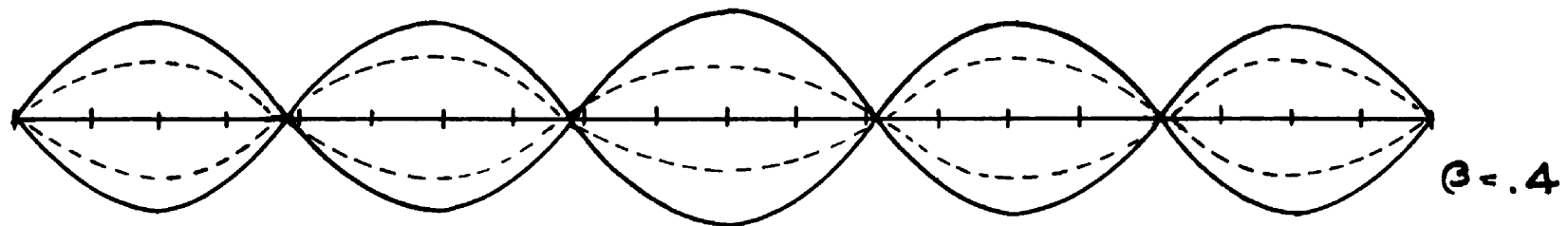
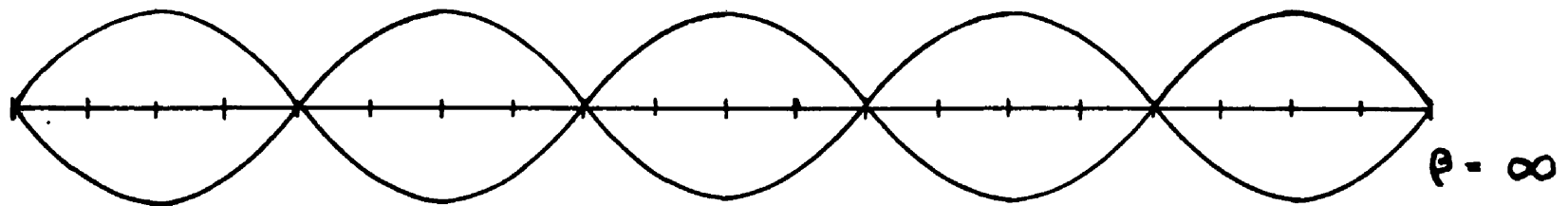


PLATE 3

one end, and simply supported at the other. This result can be checked in standard text books.

For the odd modes we merely have sinusoids as in the case of a one span simply supported beam. The frequency equation is -

$$\sin k l = 0 \quad (9)$$

Hence  $k_i = \frac{i\pi}{l} \quad i = 1, 2, 3, \dots$

Also the shape function for the odd mode is -

$$\bar{X}_i = \sin \frac{i\pi x}{l} \quad (10)$$

## 2. Use of Normal Coordinates

We employ the eigenfunctions previously derived and normal coordinates "q" to form product solutions for the non-homogeneous case at hand in the following manner -

$$y = \sum_{i=1}^{\infty} \bar{q}_i(\bar{X})_i + \sum_{i=1}^{\infty} q_i(X)_i \quad (11)$$

where  $\bar{q}$  and  $\bar{X}$  are the normal coordinates and eigenfunctions respectively for the odd modes and  $q$  and  $X$  refer to the even modes. Each series is a set of orthogonal functions (see appendix 4). It will be our task to evaluate the normal coordinates for the disturbance of the problem and to demonstrate the convergence of the series by comparison with strength of materials as well as experimental verification. The advantage of using normal coordinates as proposed is that the kinetic and potential energies of

the system will be shown to be diagonalized matrices and subsequently will yield independent equations for "q" when used in conjunction with Lagrange's equations. These equations can be directly solved and an approximate solution is effected by superposing a number of terms of (11) depending on the accuracy desired. It will be shown that only a few number of modes will be needed for engineering accuracy. The analysis henceforth will proceed in a separate but simultaneous manner for the odd and even modes. The contributions to the deflections of the even and odd modes respectively are given by the equations -

$$y = \sum_{i=1}^{\infty} q_i \left[ \sin(\eta_i)(x/l) - \frac{\cos \eta_i}{\cosh \eta_i} \sinh(\eta_i)(x/l) \right] \quad (12)$$

$$\bar{y} = \sum_{i=1}^{\infty} \bar{q}_i \sin(i\pi)(x/l) \quad (13)$$

Also the time element will be handled in three installments which we will call eras. The first era is the time interval during which the load is on the first span; era two the interval when the load is on span two; and finally the third era will refer to events when the load has completely traversed and is off the beam.

### 3. Development of The Equations of Motion in Terms of The Normal Coordinates

Kinetic and potential energy expressions will first be determined for the even modes. Starting with kinetic energy we have for our symmetrical problem the expression -

$$T = \left( \frac{A\gamma}{g} \right) \int_0^l (\dot{y})^2 dx$$

Differentiating (12) with respect to time and substituting into the above expression we have -

$$T = \frac{A\gamma}{g} \int_0^l \left[ \sum_{k=1}^{\infty} (\dot{q})_k^2 \left( \sin(\eta)_k(x/l) - \frac{\cos \eta_k \sinh(\eta)_k(x/l)}{\cosh \eta_k} \right)^2 \right] dx$$

Because of homogeneous end conditions the shape functions are orthogonal over the interval and all cross terms will disappear on integration.

Deleting these we must integrate the expression -

$$T = \frac{A\gamma}{g} \sum_{k=1}^{\infty} (\dot{q})_k^2 \int_0^l \left[ \left( \sin^2(\eta)_k(x/l) - 2 \frac{\cos \eta_k \sin(\eta)_k(x/l) \sinh(\eta)_k(x/l)}{\cosh \eta_k} + \left( \frac{\cos \eta_k}{\cosh \eta_k} \right)^2 \sinh^2(\eta)_k(x/l) \right) \right] dx \quad (14)$$

Carrying out this integration we arrive at -

$$T = \frac{A\gamma l}{g} \sum_{k=1}^{\infty} (\dot{q})_k^2 \frac{F_k}{2\eta_k} \quad (15)$$

$$\text{where } F_k = \left[ \eta_k - 3 \sin \eta_k \cos \eta_k + 3 \cos^3 \eta_k \tanh \eta_k - \frac{\cos^3 \eta_k (\eta_k)}{\cosh^3 \eta_k} \right] \quad (16)$$

For the odd modes we have on substituting (13) into the general Kinetic energy expression the result -

$$T = \frac{A\gamma l}{2g} \sum_{k=1}^{\infty} (\dot{\bar{q}})_k^2 \quad (17)$$

We now turn to the potential energy expression for the even modes. Considering the bending energy of the beam and the deformation of the midsupport we have -

$$V = (EI) \int_0^l \left[ \frac{d^2 y}{dx^2} \right]^2 dx + \frac{1}{2} K y_{(x=l)}^2 \quad (16)_a$$

Substituting from (12) into the above expression gives -

$$V = EI \int_0^l \left[ \sum_1^{\infty} q_i \frac{\gamma_i^2}{l^2} \left( -\sin(\gamma_i)(x/l) - \frac{\cos \gamma_i}{\cosh \gamma_i} \sinh(\gamma_i)(x/l) \right) \right]^2 dx + \frac{1}{2} K \left[ \sum_1^{\infty} q_i \left( \sin \gamma_i - \cos \gamma_i \tanh \gamma_i \right) \right]^2$$

Now it is shown in the appendix 5 that again the cross terms will cancel out. Therefore, integrating the squared terms we get -

$$V = \frac{EI}{2l^3} \sum_{i=1}^{\infty} (q_i)^2 \gamma_i^3 G_i + \frac{K}{2} \sum_{i=1}^{\infty} (q_i)^2 (H)_i \quad (18)$$

$$\text{where } G_i = \left[ \gamma_i + \sin \gamma_i \cos \gamma_i - \cos^2 \gamma_i \tanh \gamma_i - \frac{\cos^2 \gamma_i}{\cosh^2 \gamma_i} (\gamma_i) \right] \quad (19)$$

$$\text{and } H_i = \left[ \sin^2 \gamma_i - 2 \sin \gamma_i \cos \gamma_i \tanh \gamma_i + \cos^2 \gamma_i \tanh^2 \gamma_i \right] \quad (20)$$

In the odd modes we are not concerned with the elasticity of the midsupport and consequently the potential energy expression contains only the effect of bending of the beam.

$$\bar{V} = (EI) \sum_1^{\infty} (\bar{q}_i)^2 (i\pi/l)^4 \int_0^l \sin^2(i\pi x/l) dx$$

Integrating we have -

$$\bar{V} = \frac{EI}{2l^3} \sum_1^{\infty} (\bar{q}_i)^2 (i\pi)^4 \quad (21)$$

We can now substitute the expressions for kinetic and potential energies into Lagrange's equations,

$$\frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{q}_i} \right] + \frac{\partial V}{\partial q_i} - \frac{\partial T}{\partial q_i} = Q_i \quad (22)$$

As will be shown later the resulting equations will be of the form

$$\ddot{q}_i + \alpha_i^2 q_i = Q_i \quad (23)$$

where  $\alpha_i$  and  $Q_i$  are constants and  $Q_i$  is our generalized force term.

#### 4. Form of Solution for The Generalized Coordinates

The complementary solution of (23) is -

$$q_i = A_i \sin \alpha_i t + B_i \cos \alpha_i t$$

We use the method of variation of parameters to form the particular solution. Thus, letting  $q_i = A_i(t) \sin \alpha_i t + B_i(t) \cos \alpha_i t$  we have for  $A_i(t)$  and  $B_i(t)$  the following expressions:

$$\frac{\partial A_i(t)}{\partial t} = \frac{\begin{vmatrix} 0 & \cos \alpha_i t \\ Q_i & -\alpha_i \sin \alpha_i t \end{vmatrix}}{\begin{vmatrix} \sin \alpha_i t & \cos \alpha_i t \\ \alpha_i \cos \alpha_i t & -\alpha_i \sin \alpha_i t \end{vmatrix}}$$

$$\frac{\partial B_i(t)}{\partial t} = \frac{\begin{vmatrix} \sin \alpha_i t & 0 \\ \alpha_i \cos \alpha_i t & -\alpha_i Q_i \end{vmatrix}}{\begin{vmatrix} \sin \alpha_i t & \cos \alpha_i t \\ \alpha_i \cos \alpha_i t & -\sin \alpha_i t \end{vmatrix}}$$



Integrating the resulting expressions we have -

$$\begin{aligned} A_i(t) &= \frac{Q_i}{\alpha_i} \int (Q_i \cos \alpha_i t) dt \\ B(t) &= \frac{Q_i}{\alpha_i} \int (Q_i \sin \alpha_i t) dt \end{aligned}$$

The final form of the particular solution is therefore -

$$q_i = \frac{Q_i}{\alpha_i} \left[ \left( \int Q_i \cos \alpha_i t dt \right) (\sin \alpha_i t) - \left( \int Q_i \sin \alpha_i t dt \right) (\cos \alpha_i t) \right] \quad (24)$$

We now proceed to determine the generalized coordinates for the three eras of time previously defined. The analysis will be carried on for the even and odd modes in a separate but parallel manner.

### 5. Era One. Solution For The Even Modes

The expression for the kinetic energy as given by (15) can be expressed as -

$$T = \sum_{i=1}^{\infty} \frac{C_i}{2} (\dot{q})_i^2 \quad (25)$$

$$\text{where } C_i = \left[ \frac{A_i l}{g} \right] \left[ \frac{F_i}{\gamma_i} \right] \quad (26)$$

Also for the potential energy we have, modifying (18), -

$$V = \sum_{i=1}^{\infty} \frac{D_i}{2} (q)_i^2 \quad (27)$$

$$\text{where } D_i = \frac{EI}{l^3} \left[ \gamma^3 G_i + \frac{4}{3} H_i \right] \quad (28)$$

Substituting these expressions into the Lagrange equations we have -

$$\ddot{q}_i + \alpha_i^2 q_i = \Theta_i Q_i \quad (29)$$

where 
$$\alpha_i^2 = \left[ \frac{EI/l^3}{A^2 l/g} \right] \left[ \frac{\eta_i^3 G_i + (L/\Theta)(H)_i}{(E/\eta_i)} \right] \quad (30)$$

$$\Theta_i = \frac{(\eta_i)(g)}{(A^2 l)(F)_i} \quad (31)$$

$$\frac{\Theta_i}{\alpha_i^2} = \frac{l^3}{EI} \left[ \frac{1}{(\eta_i)^3 G_i + (L/\Theta) H_i} \right] \quad (32)$$

We now determine the generalized force term. Since  $q$  is independent of the other generalized coordinates we can give it an increment,  $\delta q_i$ . Work will be performed by the load  $P$ , which for the first era will be somewhere on span one at a point which we will call  $x_0$ . With this in mind we have on equating energies -

$$Q_i \delta q_i = P \left[ \left( \sin(\eta_i)(x_0/l) - \frac{\cos \eta_i \sinh(\eta_i)(x_0/l)}{\cosh \eta_i} \right) \right] \delta q_i$$

Since  $x_0$  equals  $vt$ , where  $v$  is the uniform velocity of the load, we have for the generalized force -

$$Q_i = P \left[ \left( \sin(\eta_i)(vt/l) - \frac{\cos \eta_i \sinh(\eta_i)(vt/l)}{\cosh \eta_i} \right) \right] \quad (33)$$

Solving for the particular solution in dimensionless form gives -

$$(q)_p = -\frac{P \Theta_i}{\alpha_i^2} \left[ \frac{\sin(\eta_i)(vt/l)}{(\omega/\alpha_i)^2 - 1} + \frac{\cos \eta_i \sinh(\eta_i)(vt/l)}{\cosh \eta_i (\omega/\alpha_i)^2 + 1} \right] \quad (34)$$

where 
$$\frac{v \eta_i}{l} = \omega_i,$$

Placing the conditions of zero initial velocity and displacement on the problem we find for the complementary solution-

$$(q_i)_c = \frac{P \Theta_i \omega_i}{\alpha_i^3} \left[ \frac{1}{(\omega/\alpha)_i^2 - 1} + \frac{\cos \gamma_i}{\cosh \gamma_i} \frac{1}{(\omega/\alpha)_i^2 + 1} \right] \sin \gamma_i (\alpha/\omega)_i (vt/l) \quad (35)$$

The final statement for the generalized coordinate for era one for the even modes is given by the following expression.

$$q_i = P \left[ \frac{\Theta_i}{\alpha_i^3} \left\{ \left[ \frac{1}{(\omega/\alpha)_i^2 - 1} + \frac{\cos \gamma_i}{\cosh \gamma_i} \frac{1}{(\omega/\alpha)_i^2 + 1} \right] \left( \sin (\alpha/\omega)_i (\gamma_i) (vt/l) \right) (\omega/\alpha)_i - \left[ \frac{\sin (\gamma_i) (vt/l)}{(\omega/\alpha)_i^2 - 1} + \frac{\cos \gamma_i}{\cosh \gamma_i} \frac{\sinh (\gamma_i) (vt/l)}{(\omega/\alpha)_i^2 + 1} \right] \right\} \right] \quad (36)$$

If measurement is made in the first span we use for the even modes-

$$y_i = q_i \left[ \sin (\gamma_i) (x/l) - \frac{\cos \gamma_i}{\cosh \gamma_i} \sinh (\gamma_i) (x/l) \right] \quad (37)$$

and should the deflections on the second span be desired we have-

$$y_i = q_i \left[ \sin (\gamma_i) \frac{(l-x)}{l} - \frac{\cos \gamma_i}{\cosh \gamma_i} \sinh (\gamma_i) \frac{(l-x)}{l} \right] \quad (38)$$

## 6. Era One. Solution of The Odd Modes

The generalized force in this case is simply-

$$Q_i = P \sin(i\pi)vt \quad (39)$$

The expressions for kinetic and potential energies are respectively

$$\bar{T} = \sum_{i=1}^{\infty} \frac{J}{2} (\bar{\dot{q}})_i^2 \quad \text{where } J = \frac{A \gamma l}{g} \quad (40)$$

$$\bar{V} = \sum_{i=1}^{\infty} \frac{N_i}{2} (\bar{q})_i^2 \quad \text{where } N_i = \frac{EI(i\pi)^4}{l^3} \quad (41)$$

Substitution into Lagrange's equations yields -

$$\ddot{\bar{q}}_i + \frac{N_i}{J} (\bar{q})_i = \frac{1}{J} Q_i$$

The natural frequencies are given by the expression -

$$\omega_i^2 = \frac{N_i}{J} = \frac{EI(i\pi)^4 g}{A \gamma l^4} \quad (42)$$

The particular solution for the odd modes is given by the following expression, in which  $\bar{\omega}_i = (i\pi)(v/l)$ .

$$(\bar{q}_i)_p = - \frac{P l^3}{EI(i\pi)^4} \left[ \frac{\sin(i\pi)(vt/l)}{(\bar{\omega}/\omega)_i^2 - 1} \right] \quad (43)$$

The complementary solution for zero initial displacements and velocities is-

$$(\bar{q}_i)_c = \frac{P l^3}{EI(i\pi)^4} \left[ \frac{(\bar{\omega}/\omega)_i \sin(i\pi)(vt/l)(\omega/\bar{\omega})}{(\bar{\omega}/\omega)_i^2 - 1} \right] \quad (44)$$

The complete solution for the odd modes for era one is therefore -

$$\bar{q}_i = \frac{P l^3}{EI(i\pi)^4} \left[ \frac{(\bar{\omega}/\omega)_i \sin(i\pi)(\omega/\bar{\omega})_i(vt/l)}{(\bar{\omega}/\omega)_i^2 - 1} - \frac{\sin(vt/l)(i\pi)}{i\pi} \right] \quad (45)$$

## 7. Era Two. Solution for the Even Modes

It will be recalled that in era two the load is on span two. For convenience a new time axis will be used for this era such that at  $t = 0$  the load will be proceeding onto the second span. The generalized force in this case is given by the expression -

$$Q_i = P \left[ \sin(l-vt)(\eta_i/l) - \frac{\cos \eta_i}{\cosh \eta_i} \sinh(l-vt)(\eta_i/l) \right] \quad (46)$$

The particular solution for this case can be shown to be -

$$(q_i)_p = \frac{P \Theta_i}{\alpha_i^2} \left[ \left( \frac{\omega}{\alpha} \right)_i \left[ \frac{1}{\left( \frac{\omega}{\alpha} \right)_i^2 - 1} + \frac{\cos \eta_i}{\cosh \eta_i} \frac{1}{\left( \frac{\omega}{\alpha} \right)_i^2 + 1} \right] \sin(\eta_i)(\alpha/\omega)_i(1-vt/l) \right. \\ \left. - \left[ \frac{\sin(\eta_i)(1-vt/l)}{\left( \frac{\omega}{\alpha} \right)_i^2 - 1} + \frac{\cos \eta_i}{\cosh \eta_i} \frac{\sinh(\eta_i)(1-vt/l)}{\left( \frac{\omega}{\alpha} \right)_i^2 + 1} \right] \right] \quad (47)$$

In addition to the above motion, there will be free vibration resulting from the initial velocity and displacement of this mode at the outset of era two. The general expression for this type of motion is -

$$(q_i)_c = A_i \sin(\eta_i)(\alpha/\omega)_i(vt/l) + B_i \cos(\eta_i)(\alpha/\omega)_i(vt/l) \quad (48)$$

The initial conditions for era two are simply the final conditions for era one. That is for  $t = 0$  in era two we have -

$$(q_i)|_0 = \frac{P \Theta_i}{\alpha_i^2} \left[ \frac{\left( \frac{\omega}{\alpha} \right)_i \sin(\eta_i)(\alpha/\omega)_i - \sin(\eta_i)}{\left( \frac{\omega}{\alpha} \right)_i^2 - 1} + \frac{\cos \eta_i}{\cosh \eta_i} \frac{\left( \frac{\omega}{\alpha} \right)_i \sin(\eta_i)(\alpha/\omega)_i - \sin(\eta_i)}{\left( \frac{\omega}{\alpha} \right)_i^2 + 1} \right] \\ (\dot{q}_i)|_0 = \frac{P \Theta_i}{\alpha_i^2} \left[ \frac{\left( \frac{\omega}{\alpha} \right)_i [\cos(\eta_i)(\alpha/\omega)_i - \cos(\eta_i)]}{\left( \frac{\omega}{\alpha} \right)_i^2 - 1} + \frac{\cos \eta_i}{\cosh \eta_i} \frac{\left( \frac{\omega}{\alpha} \right)_i [\cos(\eta_i)(\alpha/\omega)_i - \cosh(\eta_i)]}{\left( \frac{\omega}{\alpha} \right)_i^2 + 1} \right]$$

Substituting these conditions into the sum  $(q_p + q_s)$  we arrive after some manipulation at the final equation for the even mode generalized coordinates for era two.

$$q_s = \frac{P \theta_s}{\alpha_s^2} \left\{ \left[ \frac{1}{(\omega/\alpha)_s^2 - 1} + \frac{\cos \eta_s}{\cosh \eta_s} \frac{1}{(\omega/\alpha)_s^2 + 1} \right] (\omega/\alpha)_s \sin(\eta_s) (\alpha/\omega)_s (1 + vt/l) - \right. \\ \left. 2 \left[ \cos(\eta_s) \right] (\omega/\alpha)_s \left[ \frac{1}{(\omega/\alpha)_s^2 + 1} + \frac{1}{(\omega/\alpha)_s^2 - 1} \right] \sin(\eta_s) (\alpha/\omega)_s (vt/l) - \left[ \frac{\sin(\eta_s) (1 - vt/l)}{(\omega/\alpha)_s^2 - 1} \right] - \right. \\ \left. \left[ \frac{\cos \eta_s}{\cosh \eta_s} \frac{\sinh(\eta_s) (1 - vt/l)}{(\omega/\alpha)_s^2 + 1} \right] \right\} \quad (49)$$

For investigation on span one we use for  $X_s$  -

$$X_s = \left[ \sin(\eta_s) (x/l) - \frac{\cos \eta_s \sinh(\eta_s) (x/l)}{\cosh \eta_s} \right]$$

and when calculations concern span one, we have -

$$X_s = \left[ \sin(l-x) (\eta_s/l) - \frac{\cos \eta_s \sinh(l-x) (\eta_s/l)}{\cosh \eta_s} \right]$$

### 8. Era Two. Solution for The Odd Modes

Again as in the preceding case, we start a new time axis. The generalized force for this case comes out to be -

$$\bar{Q}_s = P \cos(i\pi) \sin(l-vt) (i\pi/l) \quad (50)$$

The particular and complementary solutions are given respectively as -

$$(\bar{Q}_s)_p = P \cos(i\pi) \frac{l^3}{EI (i\pi)^4} \left[ \frac{(\bar{\omega}/\bar{\alpha})_s \sin(i\pi) (\bar{\alpha}/\bar{\omega})_s (vt/l) - \sin(i\pi) (vt/l)}{(\bar{\omega}/\bar{\alpha})_s^2 - 1} \right] \quad (51)$$

$$(\bar{q}_i)_c = A_i \sin(i\pi)(vt/l)(\alpha/\bar{\omega})_i + B_i \cos(i\pi)(\alpha/\bar{\omega})_i (vt/l) \quad (52)$$

The initial conditions for era two are -

$$\begin{aligned} \bar{q}_i|_0 &= \frac{P l^3}{EI(i\pi)^4} \left[ \frac{(\bar{\omega}/\alpha)_i \sin(i\pi)(\alpha/\bar{\omega})}{(\bar{\omega}/\alpha)_i^2 - 1} \right] \\ \dot{\bar{q}}_i|_0 &= \frac{P l^3 \bar{\omega}_i}{EI(i\pi)^4} \left[ \frac{\cos(i\pi)(\alpha/\bar{\omega})_i - \cos(i\pi)}{(\bar{\omega}/\alpha)_i^2 - 1} \right] \end{aligned}$$

Substituting into  $(\bar{q}_p + \bar{q}_s)_i$  we arrive at the final result for era two for the odd modes.

$$\bar{q}_i = \frac{P l^3}{EI(i\pi)^4} \left[ \frac{(\bar{\omega}/\alpha)_i \sin(i\pi)(\alpha/\bar{\omega})(1+vt/l)}{(\bar{\omega}/\alpha)_i^2 - 1} - \frac{\cos(i\pi) \sin(i\pi)(vt/l)}{(\bar{\omega}/\alpha)_i^2 - 1} \right] \quad (53)$$

For both spans we use for  $X_i$  -  $X_i = \sin(i\pi)(x/l)$

### 9. Era Three. Solution for The Even Modes

A new time axis is again chosen such that at  $t=0$  the load is just leaving the beam. The ensuing motion is simply free vibration the equation of which is given by expression (48). The initial conditions for this mode are -

$$\begin{aligned} (q)_0 &= \frac{P Q}{\alpha_i^2} \left\{ \left[ \frac{1}{(\bar{\omega}/\alpha)_i^2 - 1} + \frac{\cosh \gamma_i}{\cosh \gamma_i} \frac{1}{(\bar{\omega}/\alpha)_i^2 + 1} \right] (\bar{\omega}/\alpha)_i \sin 2(\gamma_i)(\alpha/\bar{\omega})_i \right. \\ &\quad \left. - \frac{2\bar{\omega}_i}{\alpha_i} \left[ \frac{\cos(\gamma_i)}{(\bar{\omega}/\alpha)_i^2 - 1} + \frac{\cosh(\gamma_i)}{(\bar{\omega}/\alpha)_i^2 + 1} \right] \sin(\gamma_i)(\alpha/\bar{\omega})_i \right\} \\ (\dot{q})_0 &= \frac{P Q \bar{\omega}_i}{\alpha_i^2} \left\{ \left[ \frac{1}{(\bar{\omega}/\alpha)_i^2 - 1} + \frac{\cosh \gamma_i}{\cosh \gamma_i} \frac{1}{(\bar{\omega}/\alpha)_i^2 + 1} \right] \cos 2(\gamma_i)(\alpha/\bar{\omega})_i - \right. \end{aligned}$$

$$2 \left[ \frac{\cos(\eta)_i}{(\omega/\alpha)_i^2 - 1} + \frac{\cosh(\eta)_i}{(\omega/\alpha)_i^2 + 1} \right] \cos(\eta)_i (\alpha/\omega)_i + \frac{1}{(\omega/\alpha)_i^2 + 1} + \frac{1}{(\omega/\alpha)_i^2 - 1} \Bigg\}$$

Employing these conditions we arrive at the final statement for the even modes for era three.

$$q_i = \frac{2P\ell}{\alpha_i^3} (\omega/\alpha)_i \left[ \frac{\cos(\eta)_i (\alpha/\omega)_i - \cos(\eta)_i}{(\omega/\alpha)_i^2 - 1} + \frac{\cos \eta_i \cos(\eta)_i (\alpha/\omega)_i - \cosh(\eta)_i}{\cosh \eta_i (\omega/\alpha)_i^2 + 1} \right] \left[ \sin(\eta)_i (\alpha/\omega)_i (1 + vt/\ell) \right] \quad (54)$$

#### 10. Era Three. Solution for The Odd Modes

The initial conditions for this era are -

$$\begin{aligned} \bar{q}_i|_0 &= \frac{P \ell^3}{EI(i\pi)^4} \left[ \frac{1}{(\bar{\omega}/\alpha)_i^2 - 1} \right] \left[ (\bar{\omega}/\alpha)_i \sin(i\pi) (\alpha/\bar{\omega}) 2 \right] \\ \bar{\dot{q}}_i|_0 &= \frac{P \ell^3}{EI(i\pi)^4} \left[ \frac{1}{(\bar{\omega}/\alpha)_i^2 - 1} \right] \left[ \cos 2(i\pi) (\bar{\alpha}/\bar{\omega}) - 1 \right] \end{aligned}$$

Using these conditions and starting a new time axis we have as the final result for the odd modes of era three -

$$\bar{q}_i = \frac{P \ell^3}{EI(i\pi)^4} \left[ \frac{(\bar{\omega}/\alpha)_i}{(\bar{\omega}/\alpha)_i^2 - 1} \right] \left[ \sin(i\pi) (\alpha/\bar{\omega})_i (2 + vt/\ell) - \sin(i\pi) (\bar{\alpha}/\bar{\omega}) (vt/\ell) \right] \quad (55)$$



Even Modes

$$y = P \sum_{i=1}^{\infty} \frac{1}{\alpha_i^3} \left\{ \left[ \frac{1}{(\omega/\alpha)_i^2 - 1} + \frac{\cos \eta_i}{\cosh \eta_i} \frac{1}{(\omega/\alpha)_i^2 + 1} \right] \left[ (\omega/\alpha)_i \sin(\alpha/\omega)_i (\eta_i)(vt/l) \right] - \left[ \frac{\sin(\eta_i)(vt/l)}{(\omega/\alpha)_i^2 - 1} + \frac{\cos \eta_i \sinh(\eta_i)(vt/l)}{\cosh \eta_i (\omega/\alpha)_i^2 + 1} \right] \right\} \left\{ \begin{array}{l} \left[ \sin(\eta_i)(x/l) - \frac{\cos \eta_i \sinh(\eta_i)(x/l)}{\cosh \eta_i} \right] \quad (\text{Span 1}) \\ \left[ \sin(\eta_i)(l-x)/l - \frac{\cos \eta_i \sinh(\eta_i)(l-x)/l}{\cosh \eta_i} \right] \quad (\text{Span 2}) \end{array} \right.$$

Odd Modes

$$\bar{y} = \frac{P l^3}{EI} \sum_{i=1}^{\infty} \frac{1}{(i\pi)^3} \left[ \frac{(\omega/\alpha)_i \sin(i\pi)(\alpha/\omega)_i (vt/l)}{(\omega/\alpha)_i^2 - 1} - \frac{\sin(vt/l)(i\pi)}{\sin(i\pi)(x/l)} \right]$$

Even Modes

$$y = P \sum_{i=1}^{\infty} \frac{Q_i}{\alpha_i^3} \left\{ \left[ \frac{1}{(\omega/\alpha)_i^2 - 1} + \frac{\cos \eta_i}{\cosh \eta_i} \frac{1}{(\omega/\alpha)_i^2 + 1} \right] \left[ (\omega/\alpha)_i \sin(\eta_i) (\alpha/\omega)_i (1 + vt/l) \right] - 2(\omega/\alpha)_i \cos \eta_i \left[ \frac{1}{(\omega/\alpha)_i^2 + 1} + \frac{1}{(\omega/\alpha)_i^2 - 1} \right] \right. \\ \left. \left[ \sin(\eta_i) (\alpha/\omega)_i (vt/l) \right] - \left[ \frac{\sin(\eta_i) (1 + vt/l)}{(\omega/\alpha)_i^2 - 1} \right] - \left[ \frac{\cos \eta_i \sinh(\eta_i) (1 - vt/l)}{\cosh \eta_i (\omega/\alpha)_i^2 + 1} \right] \right\} \begin{cases} \left[ \sin(\eta_i) (x/l) - \frac{\cos \eta_i \sinh(\eta_i) (x/l)}{\cosh \eta_i} \right] & (\text{Span 1}) \\ \left[ \sin(\eta_i) (l-x)/l - \frac{\cos \eta_i \sinh(\eta_i) (l-x)/l}{\cosh \eta_i} \right] & (\text{Span 2}) \end{cases}$$

Odd Modes

$$\bar{y} = \frac{P l^3}{EI} \sum_{i=1}^{\infty} \frac{1}{(i\pi)^3} \left[ \frac{(\omega/\alpha)_i \sin(i\pi) (\alpha/\omega)_i (1 + vt/l)}{(\omega/\alpha)_i^2 - 1} - \cos(i\pi) \sin(i\pi) (vt/l) \right] \left[ \sin(i\pi) (x/l) \right]$$

Even Modes

$$y = 2P \sum_1^{\infty} \frac{\Theta_i}{\alpha_i^4} (\omega/\alpha)_i \left[ \frac{\cos(\eta_i)(\alpha/\omega)_i - \cos(\eta_i)}{(\omega/\alpha)_i^2 - 1} + \frac{\cos \eta_i}{\cosh \eta_i} \frac{\cos(\eta_i)(\alpha/\omega)_i - \cosh(\eta_i)}{(\omega/\alpha)_i^2 + 1} \right]$$

$$\left\{ \begin{array}{l} \left[ \sin(\eta_i)(x/l) - \frac{\cos \eta_i}{\cosh \eta_i} \sinh(\eta_i)(x/l) \right] \quad \text{(Span 1)} \\ \left[ \sin(\eta_i)(\alpha/\omega)_i(1+vt/l) \right] \\ \left[ \sin(\eta_i)(l-x)/l - \frac{\cos \eta_i}{\cosh \eta_i} \sinh(\eta_i)(l-x)/l \right] \quad \text{(Span 2)} \end{array} \right.$$

Odd Modes

$$\bar{y} = \frac{PQ^3}{EI} \sum_1^{\infty} \frac{1}{(i\pi)^4} \frac{i}{(\bar{\omega}/\alpha)_i^4 - 1} (\bar{\omega}/\alpha)_i \left[ \sin(\eta_i)(\alpha/\bar{\omega})_i(2+vt/l) - \sin(\eta_i)(\alpha/\bar{\omega})_i(vt/l) \right] \left[ \sin(\eta_i)(x/l) \right]$$

# FINAL RESULTS FOR STRESSES ERA 1

Even Modes

$$\sigma = E \frac{Pc}{l^2} \sum_i \frac{Q_i \eta_i^2}{\alpha_i^2} \left\{ \left[ \frac{1}{(\omega/\alpha)_i^2 - 1} + \frac{\cos \eta_i}{\cosh \eta_i (\omega/\alpha)_i^2 + 1} \right] \left[ (\omega/\alpha)_i \sin(\alpha/\omega)_i (\eta_i (vt/l)) \right] - \left[ \frac{\sin(\eta_i) (vt/l)}{(\omega/\alpha)_i^2 - 1} + \frac{\cos \eta_i \sinh(\eta_i) (vt/l)}{\cosh \eta_i (\omega/\alpha)_i^2 + 1} \right] \right\} \begin{cases} \left[ \sin(\eta_i) (x/l) - \frac{\cos \eta_i \sinh(\eta_i) (x/l)}{\cosh \eta_i} \right] & \text{(Span 1)} \\ \left[ \sin(\eta_i) (l-x)/l - \frac{\cos \eta_i \sinh(\eta_i) (l-x)/l}{\cosh \eta_i} \right] & \text{(Span 2)} \end{cases}$$

Odd Modes

$$\sigma = \frac{PQc}{I} \sum_i \frac{1}{(i\eta)^2} \left[ \frac{(\omega/\eta)_i \sin(i\pi (\omega/\eta)_i (vt/l)) - \sin(vt/l) (i\pi)}{(\omega/\eta)_i^2 - 1} \right] \left[ -\sin(i\pi) (x/l) \right]$$

Even Modes

$$\sigma = \frac{PEc}{l^2} \sum_{i=1}^{\infty} \frac{\gamma_i \Theta_i}{\alpha_i^2} \left\{ \left[ \frac{1}{(\omega/\alpha)_i^2 - 1} + \frac{\cos \gamma_i}{\cosh \gamma_i} \frac{1}{(\omega/\alpha)_i^2 + 1} \right] \left[ (\omega/\alpha)_i \sin(\gamma_i) (\alpha/\omega)_i (1 + vt/l) \right] - 2(\omega/\alpha) \cos \gamma \left[ \frac{1}{(\omega/\alpha)_i^2 - 1} + \frac{1}{(\omega/\alpha)_i^2 + 1} \right] \right.$$

$$\left. \left[ \sin(\gamma_i) (\alpha/\omega) (vt/l) \right] - \left[ \frac{\sin(\gamma_i) (1 - vt/l)}{(\omega/\alpha)_i^2 - 1} \right] - \left[ \frac{\cos \gamma_i \sinh(\gamma_i) (1 - vt/l)}{\cosh \gamma_i (\omega/\alpha)_i^2 + 1} \right] \right\} \begin{cases} \left[ -\sin(\gamma) (x/l) - \frac{\cos \gamma_i \sinh(\gamma_i) (x/l)}{\cosh \gamma_i} \right] \text{ (Span 1)} \\ \left[ -\sin(\gamma_i) (l-x)/l - \frac{\cos \gamma_i \sinh(\gamma_i) (l-x)/l}{\cosh \gamma_i} \right] \text{ (Span 2)} \end{cases}$$

Odd Modes

$$\bar{\sigma} = \frac{PEc}{I} \sum_{i=1}^{\infty} \frac{1}{(i\pi)^2} \left[ \frac{(\bar{\omega}/\alpha)_i \sin(i\pi) (\bar{x}/\bar{\omega})_i (1 + vt/l)}{(\bar{\omega}/\alpha)_i^2 - 1} - \cos(i\pi) \sin(i\pi) (vt/l) \right] \left[ -\sin(i\pi) (x/l) \right]$$

# FINAL RESULTS FOR STRESSES ERA 3

Even Modes

$$\sigma = \frac{2PEc}{l^2} \sum_{i=1}^{\infty} \frac{\alpha_i^2 (w/\alpha)_i}{\alpha_i^2} \left[ \frac{\cos(\eta_i)(\alpha/w)_i - \cos(\eta_i)}{(w/\alpha)_i^2 - 1} + \frac{\cos \eta_i}{\cosh \eta_i} \frac{\cos(\eta_i)(\alpha/w)_i - \cosh(\eta_i)}{(w/\alpha)_i^2 + 1} \right]$$

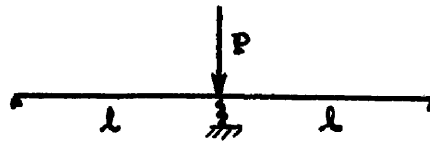
$$\left\{ \begin{array}{l} \left[ -\sin(\eta_i)(x/l) - \frac{\cos \eta_i}{\cosh \eta_i} \sinh(\eta_i)(x/l) \right] \quad (\text{Span 1}) \\ \left[ \sin(\eta_i)(\alpha/w)(1 + vt/l) \right] \\ \left[ -\sin(\eta_i)(l-x)/l - \frac{\cos \eta_i}{\cosh \eta_i} \sinh(\eta_i)(l-x)/l \right] \quad (\text{Span 2}) \end{array} \right.$$

Odd Modes

$$\bar{\sigma} = \frac{Plc}{I} \sum_{i=1}^{\infty} \frac{1}{(i\pi)^2} \frac{1}{(\bar{w}/2)_i^2 - 1} \left[ (\bar{w}/\alpha)_i \right] \left[ \sin(\eta_i)(\bar{\alpha}/\bar{w})_i(2 + vt/l) - \sin(\eta_i)(\bar{\alpha}/\bar{w})_i(vt/l) \right] \left[ -\sin(\eta_i)(x/l) \right]$$

### STATIC CHECK OF RESULTS

In order to check the accuracy and to get some idea of the rate of convergence at various parts of the beam we examine the case where  $(\omega/\alpha)$  becomes zero and  $(vt/l)$  becomes 1. This amounts to a static load at the midpoint of our beam as shown in the diagram given below.



The solution from strength of materials which may be found in the appendix 6 is given as -

$$y = \frac{3P}{2K(1.5\beta + 1)} \left[ \frac{(x/l)^3}{3} + (x/l) \right]$$

For the deflection at the midpoint of the beam we have from above -

$$y = \frac{P}{K} \left[ \frac{1}{1/5\beta + 1} \right]$$

where  $K$  is the spring constant and  $\beta$  is the dimensionless parameter used in the vibration analysis.

Now a vibrational analysis shows immediately that the odd modes contribute nothing to the deflection. Therefore the equation for the even modes for the first era is -

$$y_i = \frac{P\beta_i}{\alpha_i^3} \left[ \sin \gamma_i - \cos \gamma_i \tanh \gamma_i \right] \left[ \sin(\gamma_i)(x/l) - \cos(\gamma_i)(x/l) \tanh(\gamma_i)(x/l) \right]$$

Substituting 
$$Q_i = \frac{E}{A r l (E) / (\eta)_i}$$

and 
$$\alpha_i^2 = \frac{EI/l^3}{A r l / g} \left[ \frac{\eta_i^3 G_i + (4/\beta)(H)_i}{(E/\eta)_i} \right]$$

we get after some manipulation -

$$q_i = \frac{P}{K} \left[ \frac{4}{\beta [\eta_i^3 G_i + (4/\beta)(H)_i]} \right] [\sin \eta_i - \cos \eta_i \tanh \eta_i] [\sin \eta_i - \cos \eta_i \tanh \eta_i]$$

Let us choose an arbitrary value of  $\beta = .04213$ . Substituting into our formula from strength of materials we get -

$$y = \frac{P}{K} (.9405)$$

The first even mode gives the result -

$$y = \frac{P}{K} (.7445)$$

while the second even mode contributes -

$$y = \frac{P}{K} (.1961)$$

The sum of these two contributions is -

$$y = \frac{P}{K} (.9406)$$

This illustrates the rapid convergence of the series and also serves as somewhat of a check on the work. We next study the rate of convergence at another point - that of the midcenter of the first span. From strength



of materials we have the result -

$$y = \frac{P}{K} (.6467)$$

From the vibrational analysis we have for the first mode -

$$y = \frac{P}{K} (.7561)$$

For the first two modes we have -

$$y = \frac{P}{K} (.6596)$$

For the first four modes we have -

$$y = \frac{P}{K} (.6444)$$

One sees that, while the convergence at the midpoint of the span is not as good as at the midsupport, nevertheless, we may still consider the procedure to yield rapidly converging results.

### DISCUSSION OF THE ENVELOPE CURVES

The elasticity of the midsupport, as has already been pointed out, has no effect on the odd modes. However, there can be definite effects on the even modes depending on several factors which will presently be considered.

Following this discussion will be found a set of six pages of plots. The first three are concerned with the first even mode while the last group deals with the second even mode. On each page the elasticity is varied while the speed of traverse is held constant. The former item is denoted as  $\beta$  while the latter is taken as  $(\omega/\alpha)_s$ , as measured for the case of the rigid base. The amplitude of the deflection envelopes are plotted against the dimensionless time parameter  $(vt/l)$ , with the full lines representing the results of dynamic loads and the dotted lines representing the static loads. We will compare the ratio between the dynamic envelope to the corresponding static envelope in each case.

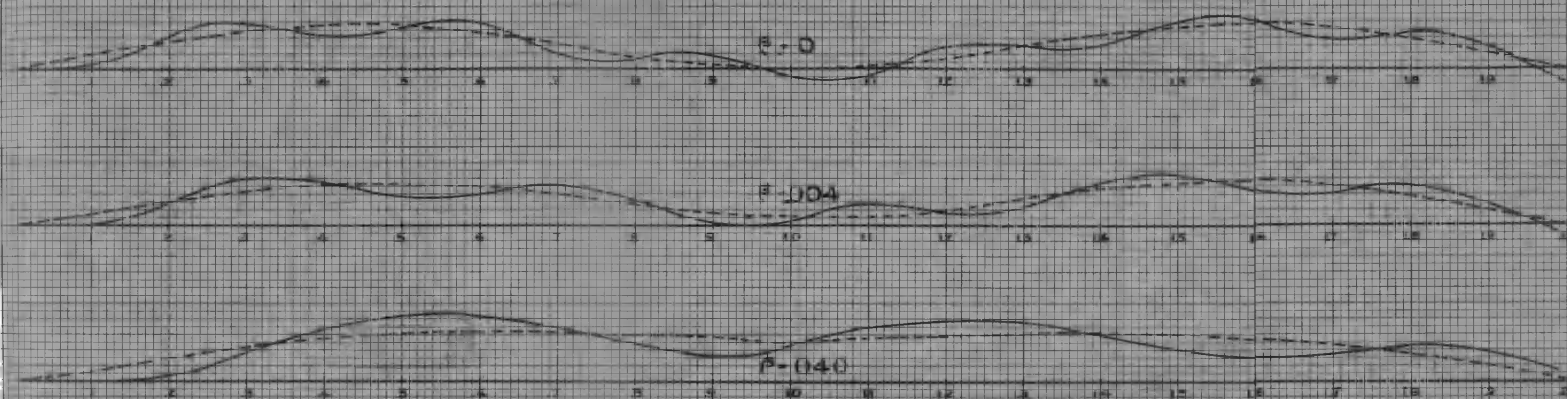
Examining the first even mode, one finds that when  $(\omega/\alpha)_s = .2$  increases in  $\beta$  (that is increasing the elasticity of the midsupport) results in very little increase in amplitude ratios. One notices mainly that since the natural frequency of the mode decreases as  $\beta$  increases, there is a slower oscillation of the dynamic envelope about the static envelope. If now, one starts with  $(\omega/\alpha)_s = .4$  on the following page a definite increase in amplitude ratio is evidenced as  $\beta$  becomes .04. It should be kept in mind as we proceed from page to page of these curves that the time axis is changing for each value of  $(\omega/\alpha)_s$ . That is a unit on the time axis of  $(\omega/\alpha)_s = .4$  actually represents half the time represented by a corresponding unit on the time axis of the  $(\omega/\alpha)_s = .2$  curves.

Proceeding to the case of  $(\omega/\alpha)_1 = .6$  one notices a definite decrease in the ratios of amplitudes when  $\beta$  is .040. Furthermore any further increase in elasticity will continue to decrease this ratio.

One can say in conclusion that for the first even mode, moderate increase in elasticity ( $\beta \rightarrow .4$ ) for cases where  $(\omega/\alpha)_2$  is about between .6 and .75 will have definite results on increasing the ratio of dynamic to static envelope ratios. Below this speed the effects are very small, while above we are on the safe side since an increase in elasticity will decrease the ratios of dynamic to static envelopes.

Examining the second mode for  $(\omega/\alpha)_2 = .2$  one finds small amplitude ratio changes just as was the case in the first even mode. However the ratios are somewhat larger and this fact is very noticeable throughout the second mode. In the cases of  $(\omega/\alpha)_2 = .4$  and .6 respectively one notices a continual growth of amplitude ratios as the elasticity is increased. Actually the case for a decrease in amplitude ratios for a moderate increase in  $\beta$  has not been reached in these curves.

We can conclude that in the second mode amplitude ratio increases are larger, appear earlier, and persist longer for moderate increase in elasticity than was the case in the first even mode. Approximately one can say that between  $(\omega/\alpha)_2 = .5$  and .85 sizeable increases in amplitude can be expected when moderate increase in elasticity is the case. Below this speed the effects of the latter are very small while above, one can expect decreases in the amplitude ratios as the elasticity is increased.



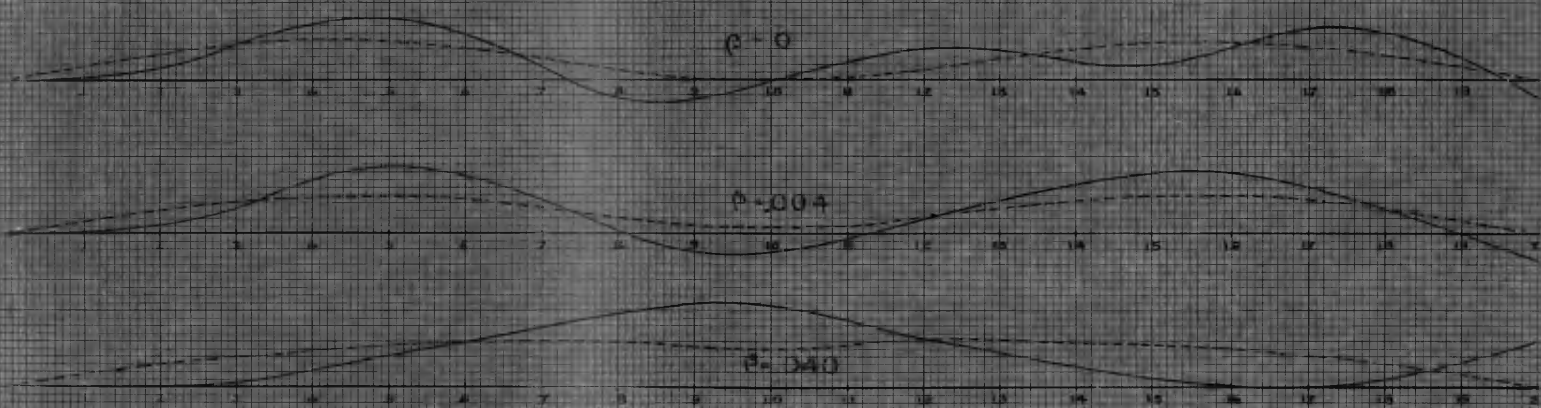
PLOTS OF RELATIVE ENVELOPE AMPLITUDES  
VS.  $V/L$  FOR THREE VALUES  $P$

FIRST EVEN MODE

$$\eta_{d1} = 2$$

STATIC LOAD - - -

DYNAMIC LOAD - - -

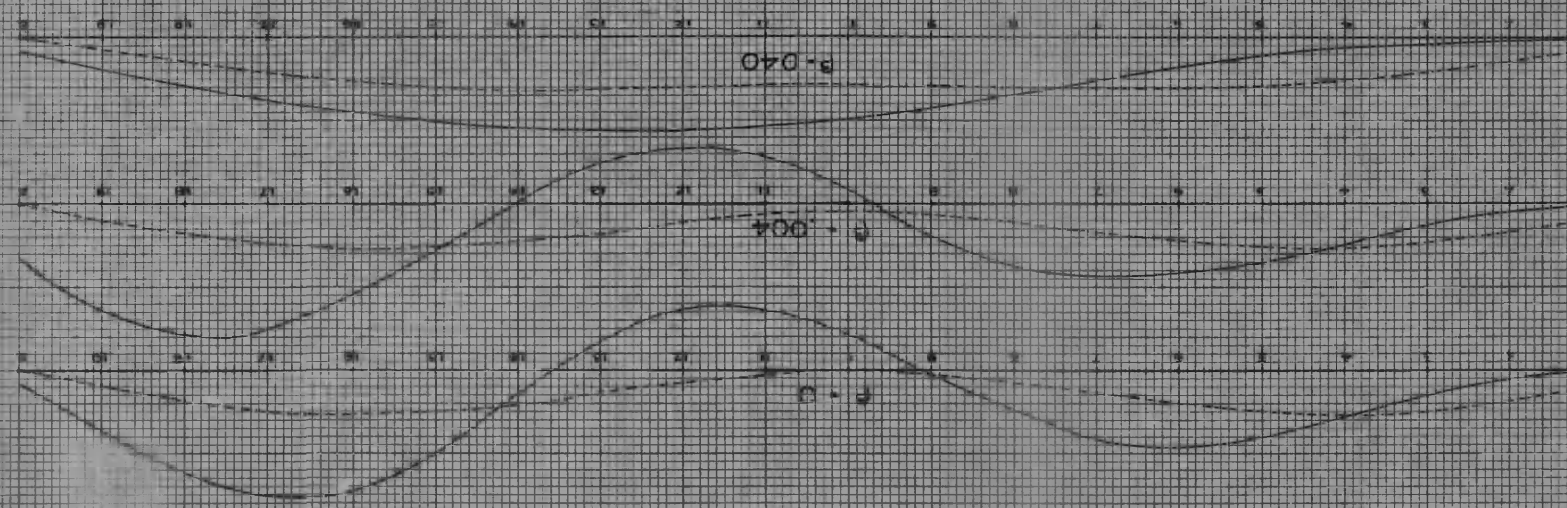


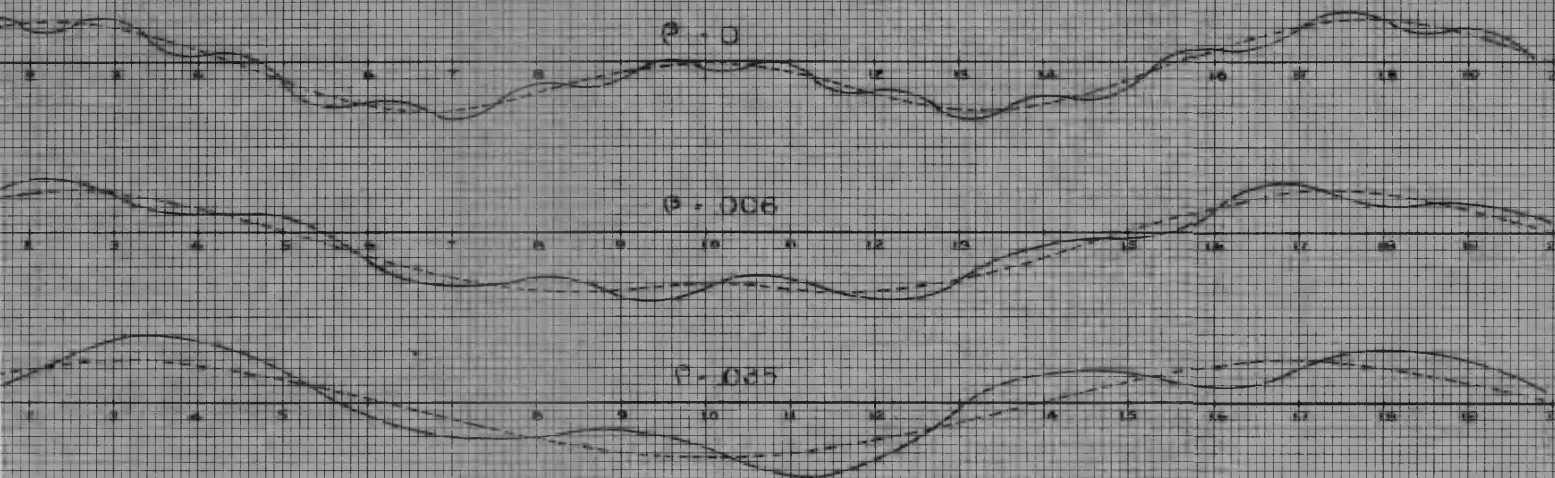
PLOTS OF RELATIVE ENVELOPE AMPLITUDES  
VS.  $x/l$  FOR THREE VALUES  $P$ .

FIRST EVEN MODE  
 $n_0 = 4$   
STATIC LOAD ---  
DYNAMIC LOAD —

FIRST EYE VIEW  
 $\frac{W}{L} = 0.6$   
 STATIC LOAD  
 DYNAMIC LOAD

PLOTS OF RELATIVE ENVELOPE AMPLITUDES  
 VS.  $\frac{W}{L}$  FOR THREE VALUES OF  $\rho$

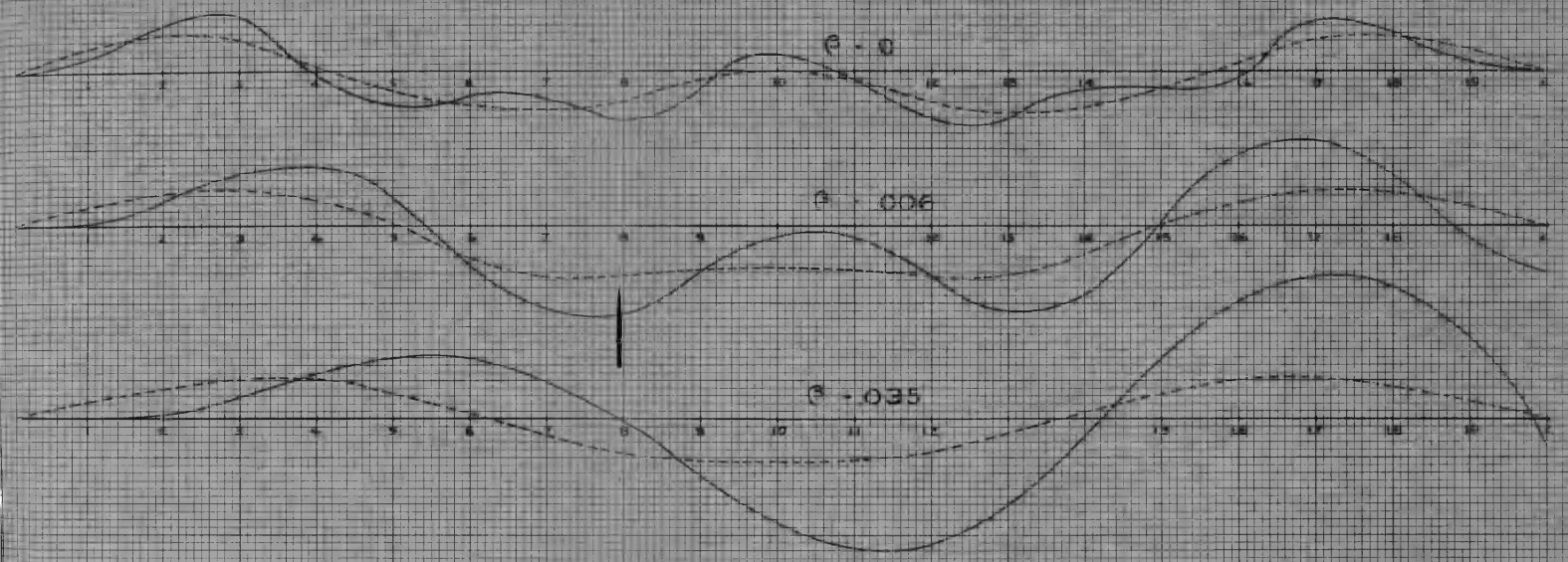




PLOTS OF RELATIVE ENVELOPE AMPLITUDES  
VS.  $V/V_L$  FOR THREE VALUES  $\beta$

SECOND EVEN MODE  
 $\frac{u_2}{u_1} = 2$   
STATIC LOAD  
DYNAMIC LOAD



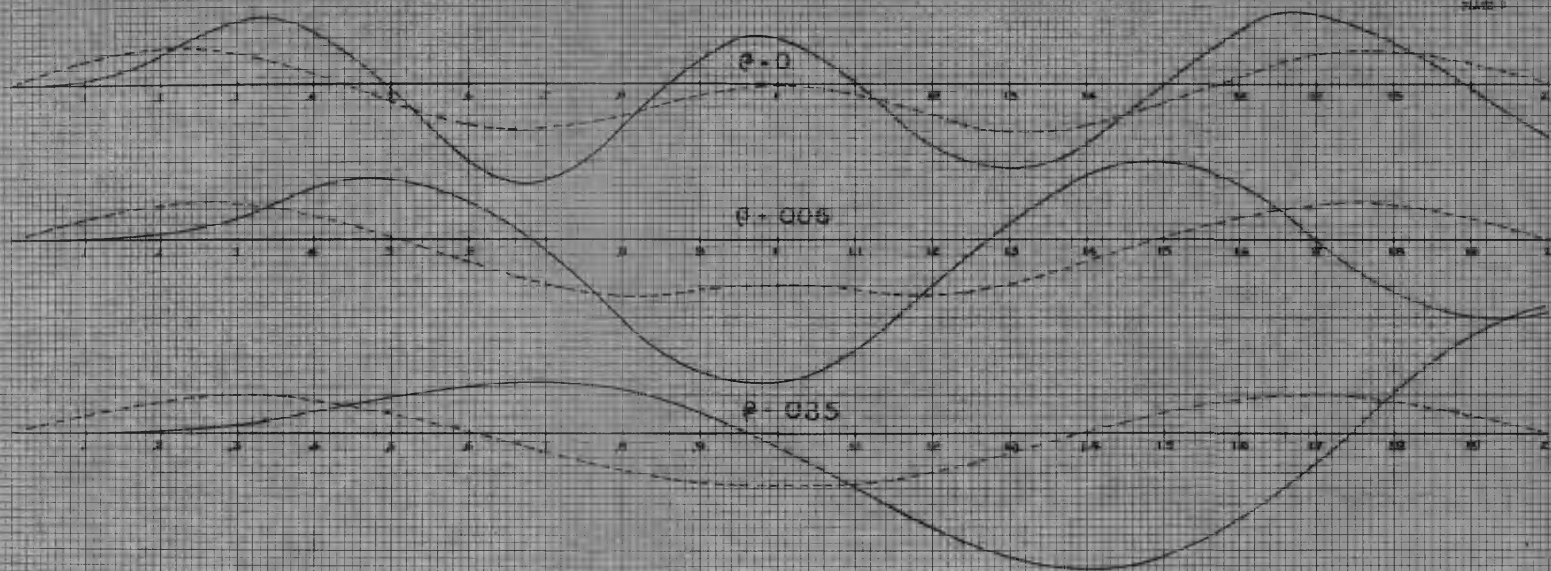


PLOTS OF RELATIVE ENVELOPE AMPLITUDES  
VS.  $V/L$  FOR THREE VALUES  $P$

SECOND EVEN MODE  
 $\frac{4\pi x_c}{L} = 4$   
STATIC LOAD - - - -  
DYNAMIC LOAD - - - -

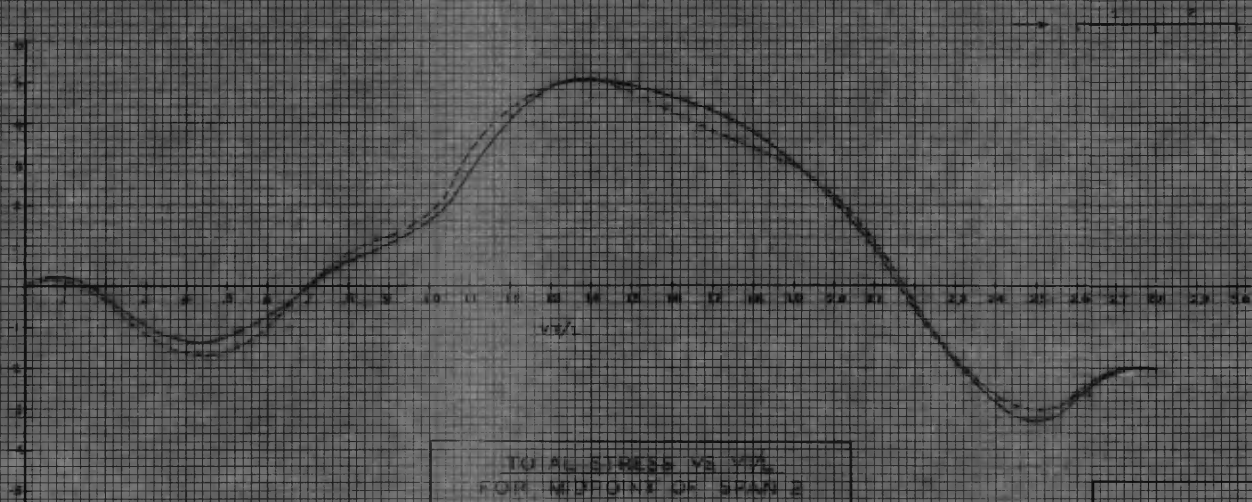


FIGURE 1



PLOTS OF RELATIVE ENVELOPE AMPLITUDES  
VS.  $y/L$  FOR TABLE VALUES  $\rho$

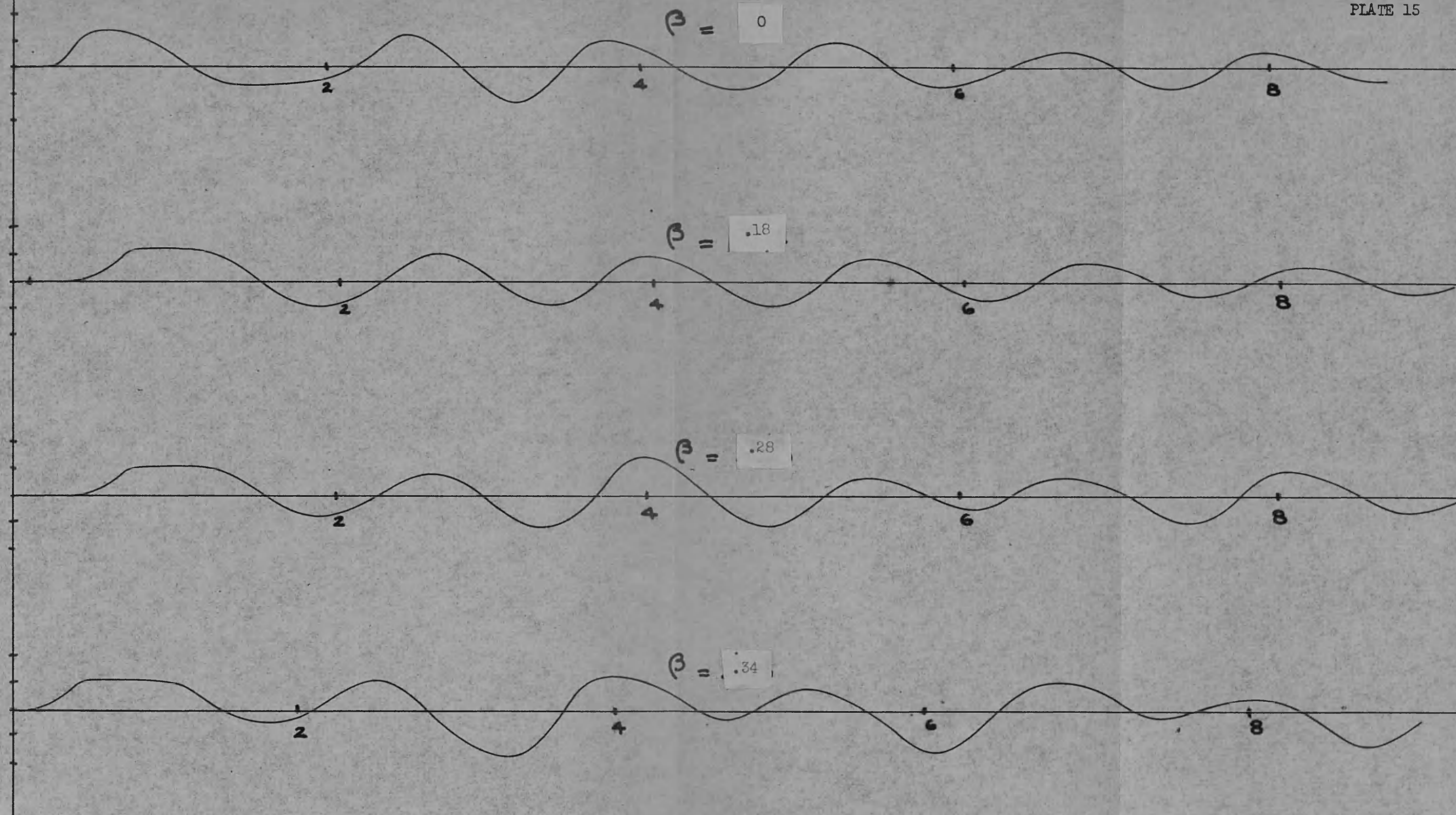
SECOND EVEN MODE  
 $\omega/\omega_c = 0$   
STATIC LOAD -----  
DYNAMIC LOAD —————



TOTAL STRESS VS  $v/l$   
 FOR MIDPOINT OF SPAN 2  
 $\frac{\sigma}{\sigma_{max}} = 1$        $\rho = 385$

EXPERIMENTAL  
 THEORETICAL  
 (4 MODES)

PLATE 15

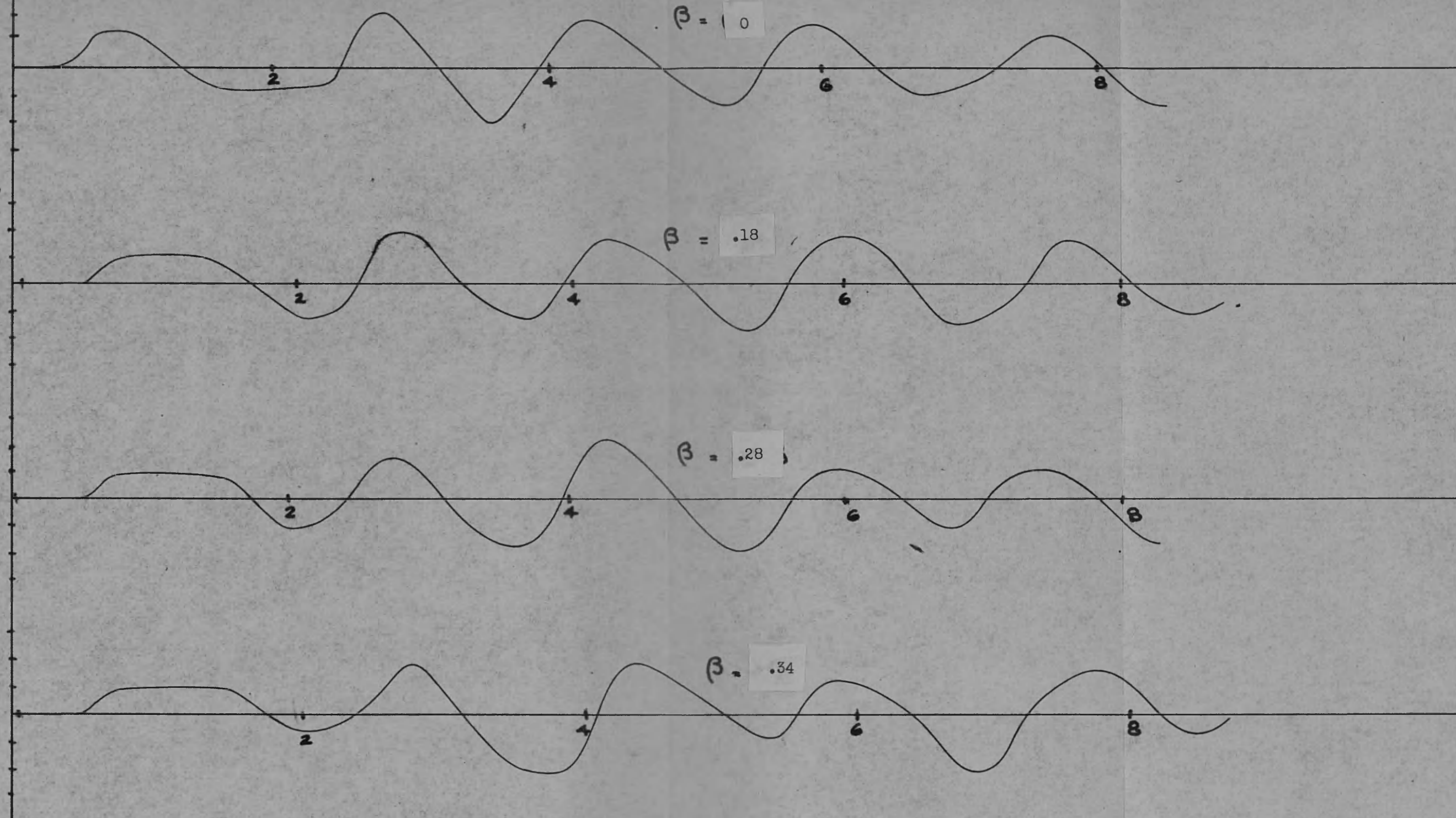


EXPERIMENTAL TOTAL STRESS  
VS.  $V.T./L$  ——— CENTER OF SPAN 1

$\bar{w}/\alpha_0 =$   
.6

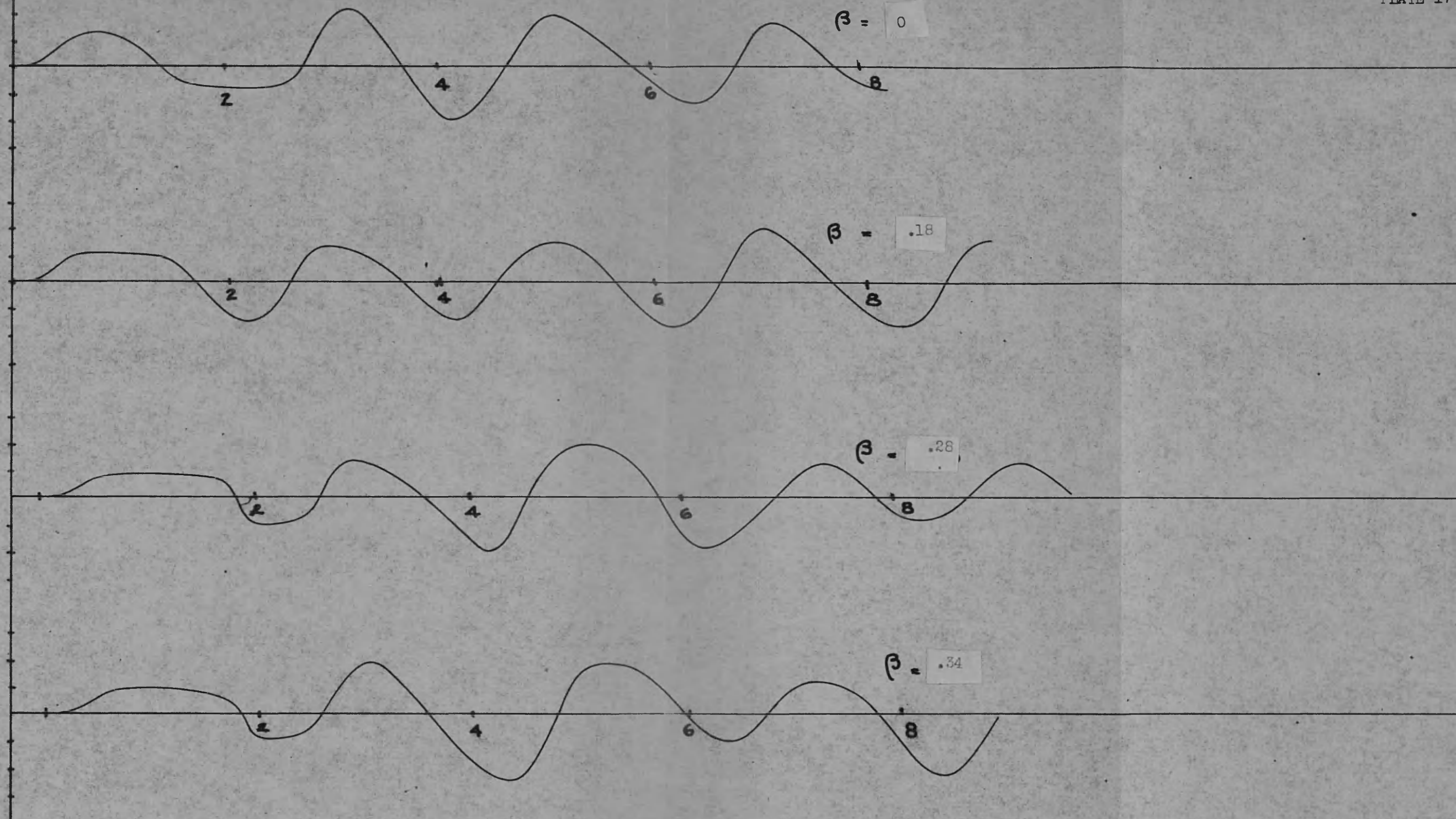


PLATE 16



EXPERIMENTAL TOTAL STRESS  
VS.  $VT/L$  ~ CENTER OF SPAN 1

$\bar{w}/\bar{\alpha} =$   
.8

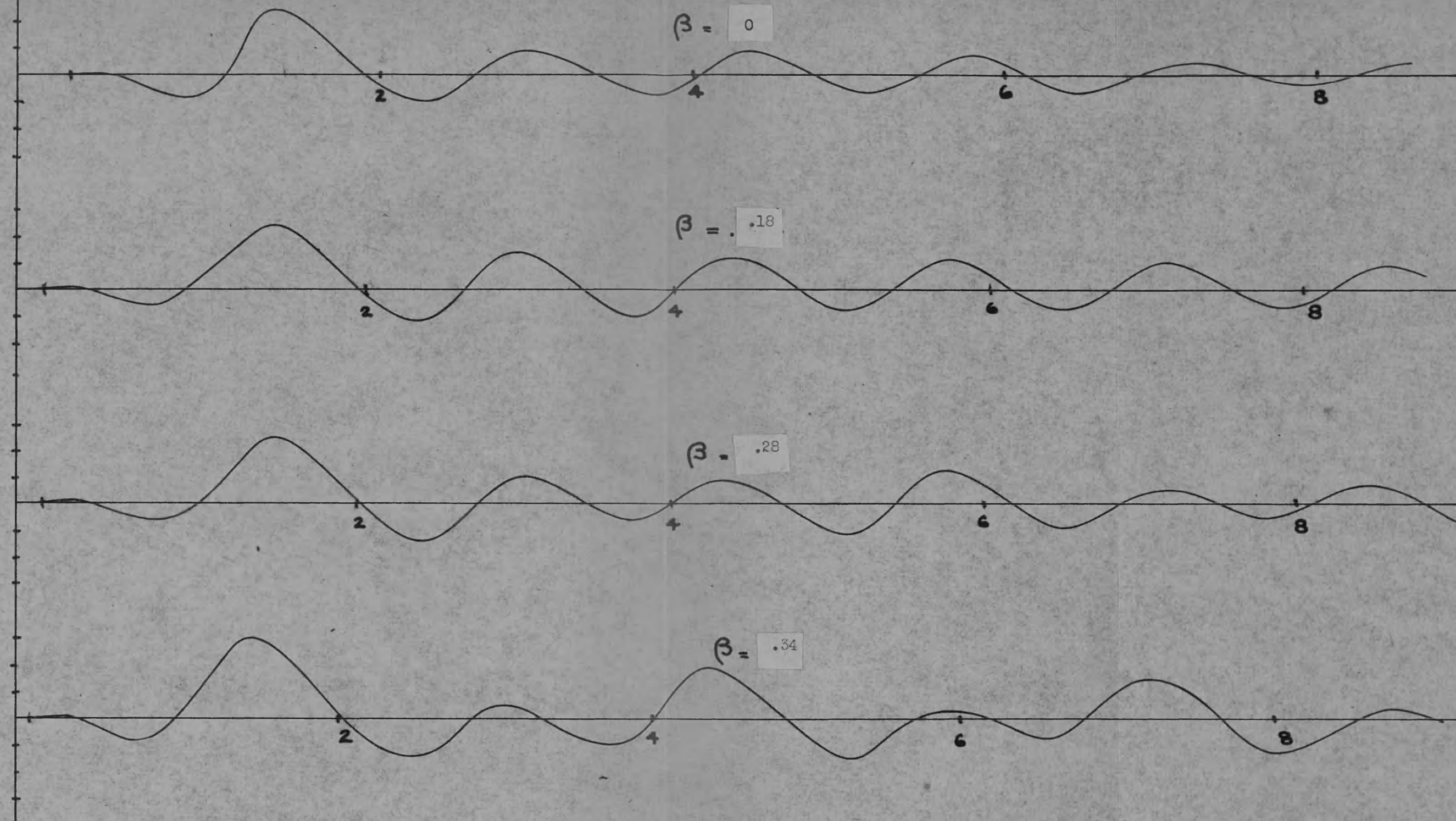


EXPERIMENTAL TOTAL STRESS  
VS.  $VT/L \sim$  CENTER OF SPAN 1

$\bar{\omega}/\alpha_1 =$   
.85



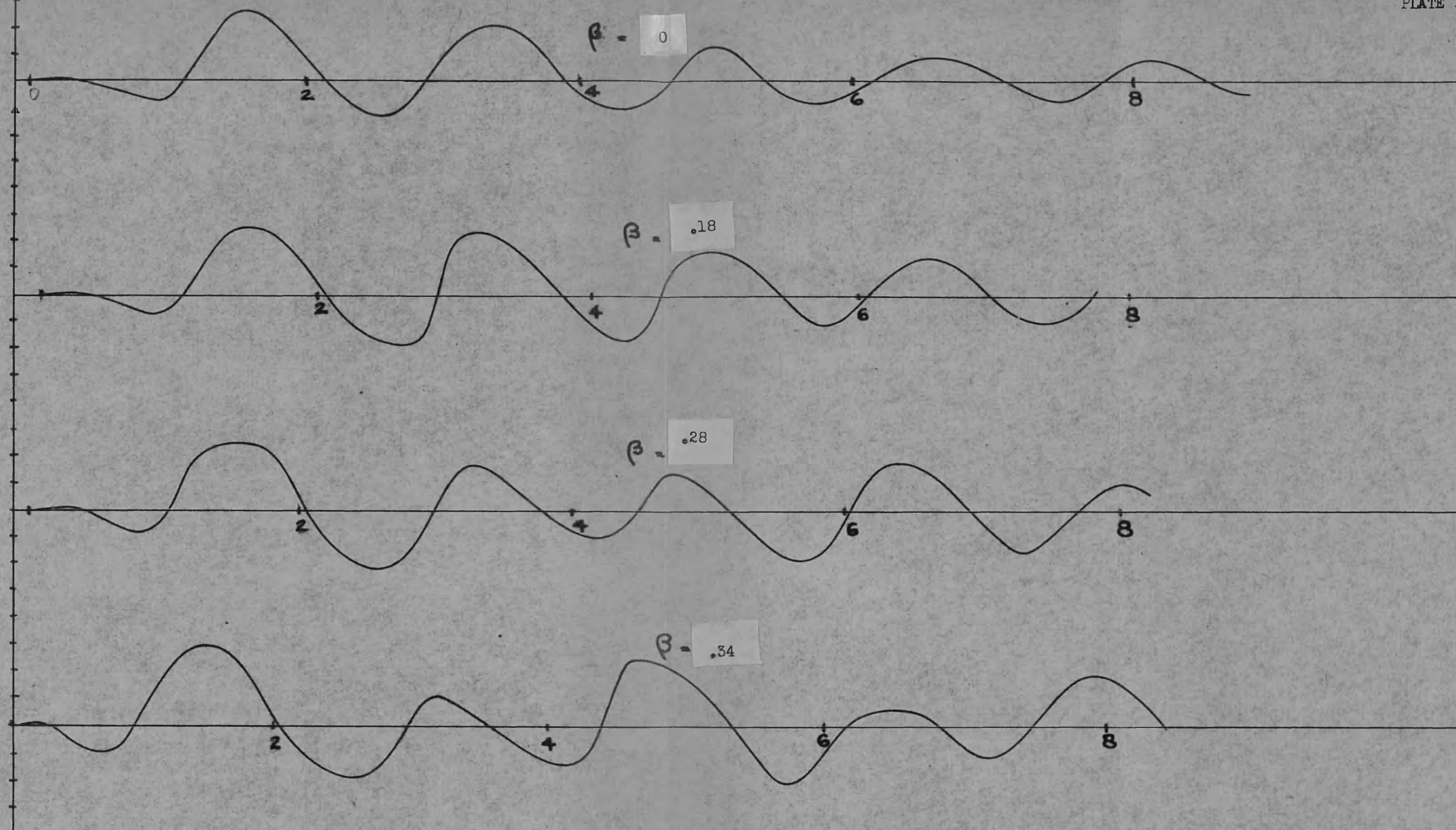
PLATE 18



EXPERIMENTAL TOTAL STRESS  
VS.  $VT/L \sim$  CENTER OF SPAN 2

$w/2 =$   
.6

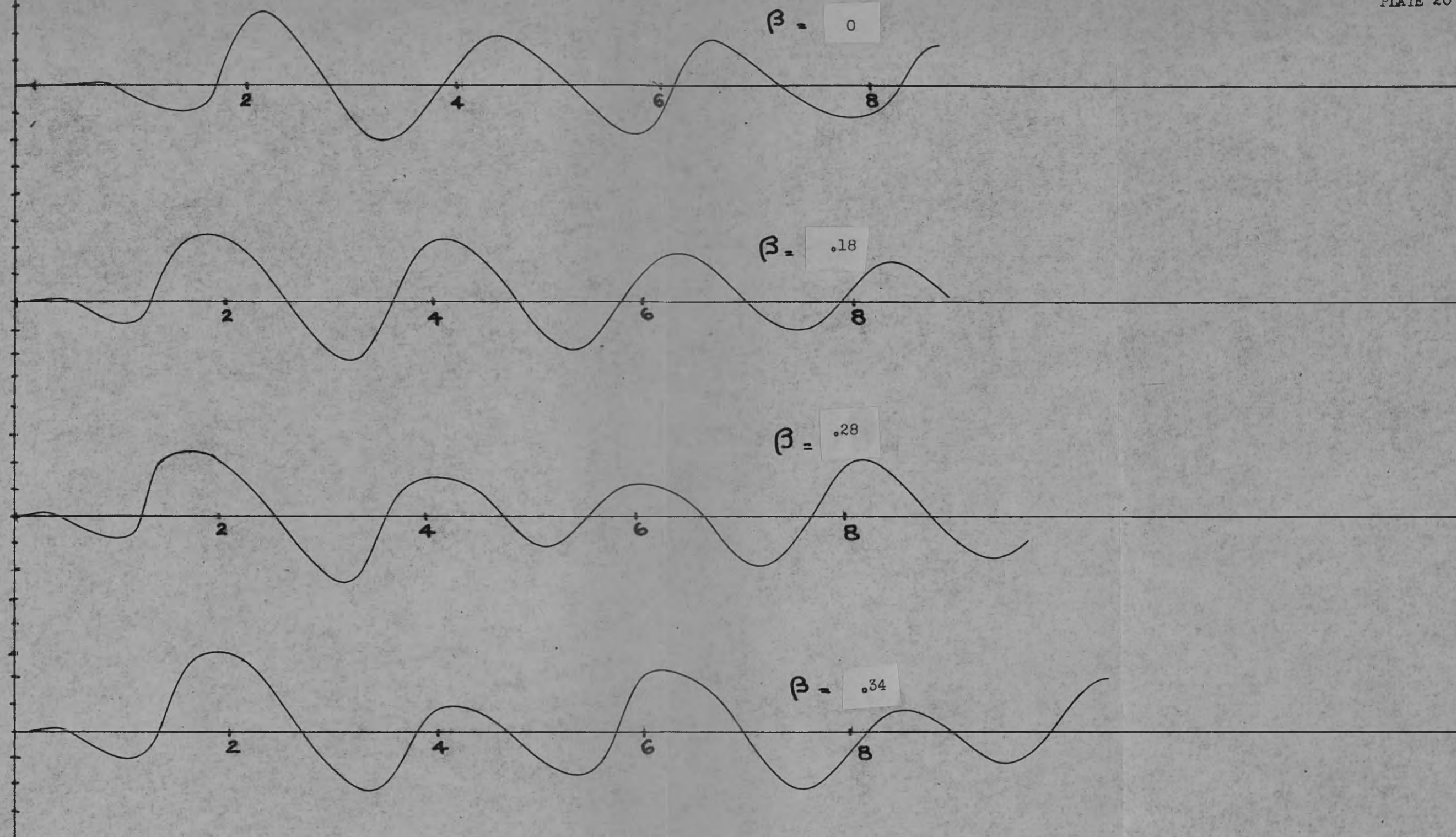
PLATE 19



EXPERIMENTAL TOTAL STRESS  
VS.  $VT/L$  ~ CENTER OF SPAN 2

$\bar{\omega}/2$   
.80





EXPERIMENTAL TOTAL STRESS  
VS.  $VT/L$  ~ CENTER OF SPAN 2

$\bar{w}/\bar{\alpha}_1$   
.85